The manuscripts should be sent to the Editor-in-Chief
Peyo Stoilov, e-mail: peyyoyo@mail.bg
Department of Mathematics, Technical University, Tsanko
Dyustabanov №25, Plovdiv, BULGARIA.
Acceptance for publication will be based on a positive
recommendation by a member of the Editorial Bord.
СПИСАНИЕ
НА ТЕХНИЧЕСКАЯ УНИВЕРСИТЕТ
В ПЛОВДИВ

<ФУНДАМЕНТАЛНИ НАУКИ
И ПРИЛОЖЕНИЯ>

VOL. 14 2009

Series A - Pure and Applied Mathematics

JOURNAL OF THE TECHNICAL UNIVERSITY AT PLOVDIV
<FUNDAMENTAL SCIENCES AND APPLICATIONS> publishes
new and original results in the fields MATHEMATICS, MECHANICS,
PHYSICS, CHEMISTRY, ECONOMICS AND THEIR
APPLICATIONS IN TECHNICAL SCIENCES.
CONTENTS

1. Young's type inequalities and some their applications
   Peyo Stoilov, Todor Kostadnov
   Page 7

2. An improperly posed problem for the one-dimensional heat equation
   Valentina Proytcheva
   Page 13

3. The exact distribution of the ump test for diagonality of covariance matrices with missing elements
   Evelina Veleva
   Page 19

4. Uniformly distributed positive definite matrices with bounded trace
   Evelina Veleva
   Page 27

5. Linear integral inequalities involving "maxima"
   S. S. Gluhcheva, K. V. Stefanova
   Page 35

6. Boundary value problem for a class of fourth-order partial differential equations of mixed type
   Georgi P. Paskalev
   Page 43
7. Smoothness of the solutions of boundary value problem for a class of fourth-order partial differential equations of mixed type

Georgi P. Paskalev

8. On isotropic compositions in pseudo–Weyl spaces

Ivan Badev
Young’s type inequalities and some their Applications

Peyo Stoilov, Todor Kostadinov

Abstract

In the present a methodical note we prove:
Let \( f(x) \geq 0 \) is differentiable function for \( x \in [0, p] \), \( y = f'(x) \),
\( f'(0) = 0 \) is continuous and a strictly increasing function for \( x \in [0, p] \),
\( x = \varphi(y) \) is the inverse function of \( y = f'(x) \).
Then for every \( 0 < a \leq p \), \( 0 < b \leq \varphi(p) \) we have:
\[
\begin{align*}
ab & < f(a) + b \varphi(b) - f(\varphi(b)); \\
f(a) + \frac{a}{\varphi(b)} [b \varphi(b) - f(\varphi(b))] & < \ab, \text{ if } a < \varphi(b); \\
\frac{b}{f'(a)} [f(a) + b \varphi(b) - f(\varphi(b))] & < \ab, \text{ if } a > \varphi(b),
\end{align*}
\]
holds with equality if and only if \( a = \varphi(b) \).

1. Introduction

Classical Young’s inequality is as follows.

Inequality of Young

Let \( f(x) \) is continuous and a increasing function for \( x \in [0, p] \), \( f(0) = 0 \).

Then for every \( 0 < a \leq p \), \( 0 < b \leq \varphi(p) \) we have

\[
(1) \quad \ab \leq \int_0^a f(x)dx + \int_0^b g(y)dy,
\]
where \( x = \varphi(y) \) is the inverse function of \( y = f(x) \).

The inequality (1) follows from the correlation
\[
S = \ab \leq S_1 + S_2,
\]
where $S$, $S_1$, $S_2$ are represented areas on the diagram:

In particular, when $y = f(x) = x^\alpha$, $x = g(y) = y^{1/\alpha}$, $\alpha > 0$, from (1) follows:

$$ab \leq \frac{a^{\alpha+1}}{\alpha + 1} + \frac{b^{\alpha+1}}{\alpha + 1}.$$

If we put $p = \alpha + 1$, $q = \frac{\alpha + 1}{\alpha}$, then we get

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

which is valid for $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Inequality (2) has many applications. For example by (2) easily follows Holder’s inequality:

$$\left|\int_a^b u(x)v(x)\,dx\right| \leq \left(\int_a^b |u(x)|^p\,dx\right)^{1/p} \left(\int_a^b |v(x)|^q\,dx\right)^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Sometimes (2) is also referred to as Young’s inequality.

The interest for finding simple analytic proof of the Young’s inequality and also its generalizations continues even nowadays [1] – [5].

**Inequality of G. Totov**

In [1] G. Totov has proved the following inequality:

$$ab \leq \int_a^b (a - x) f(x)\,dx + \int_0^b \psi(y)\,dy,$$

where $f(x)$ is continuous and a positive function for $x \geq 0$,

$x = \psi(y)$ is the inverse function of $y = \int_0^x f(t)\,dt$.

The proof of (3) is also based on the correlation $S = ab \leq S_1 + S_2$,
Young’s type inequalities and some their applications

where the areas $S_1$, $S_2$ are bounded by the curve $y = \int_o^x f(t)dt$.

As an application of (3), in [1] are proved the inequalities:

(4) $ab - b + 1 \leq a^b$, where $a > 0$, $b > 1$;

(5) $e^a + \ln(1 + b)^{1+b} \geq ab + a + b + 1$, where $a > 0$, $b > 0$.

2. Main results
Young’s type inequalities

We propose simple analytic proofs of the following inequalities.

**Theorem.** Let $f(x) \geq 0$ is differentiable function for $x \in [0, p]$, $p > 0$, $y = f'(x)$, $f'(0) = 0$ is continuous and an increasing function for $x \in [0, p]$. $x = \varphi(y)$ is the inverse function of $y = f'(x)$.

Then for every $0 < a \leq p$, $0 < b \leq \varphi(p)$ we have:

(6) $ab < f(a) + b\varphi(b) - f(\varphi(b))$;

(7) $f(a) + \frac{a}{\varphi(b)}[b\varphi(b) - f(\varphi(b))] < ab$, if $a < \varphi(b)$;

(8) $\frac{b}{f'(a)}f(a) + b\varphi(b) - f(\varphi(b)) < ab$, if $a > \varphi(b)$.

Equality in (6), (7) and (8) we have if and only if $a = \varphi(b)$.

**Proof.** Let $0 < a \leq \varphi(b)$.

We apply the Lagrange’s formula for $f(x)$ in the interval $[a, (b)]$:

$f(\varphi(b)) - f(a) = f'(c)(\varphi(b) - a)$, where $a < c < \varphi(b)$.

If $f'(c) = m$, then $f'(a) < m < b$ and hence

$f(\varphi(b)) - f(a) \leq b(\varphi(b) - a) = b\varphi(b) - ab \Rightarrow ab < f(a) + b\varphi(b) - f(\varphi(b))$, 
from which follows (6). The same inequality follows when \( a \geq \varphi(b) \).

In order to prove (7), let \( a < \varphi(b) \).

As \( f(x) \) is a convex function, then

\[
\frac{f(a)}{a} < \frac{f(\varphi(b))}{\varphi(b)} \iff f(a) < a \frac{f(\varphi(b))}{\varphi(b)}
\]

\[
\iff f(a) + ab < ab + a \frac{f(\varphi(b))}{\varphi(b)} \iff f(a) + a \frac{b\varphi(b) - f(\varphi(b))}{\varphi(b)} < ab.
\]

Let now \( a > \varphi(b) \Rightarrow f'(a) > f'(\varphi(b)) = b \).

We put \( g(y) = y\varphi(y) - f(\varphi(y)) \). Since

\[
g'(y) = \varphi(y) + y\varphi'(y) - f'(\varphi(y))\varphi'(y) = \\
= \varphi(y) + y\varphi'(y) - y\varphi'(y) = \varphi(y),
\]

then \( g(y) \) is a convex function and if \( f'(a) > b \), then

\[
\frac{g(b)}{b} < \frac{g(f'(a))}{f'(a)} \iff g(b) < f'(a)a - f(a)
\]

\[
\iff g(b) < ab - \frac{b}{f'(a)} f(a) \iff \frac{b}{f'(a)} f(a) + g(b) < ab
\]

\[
\iff \frac{b}{f'(a)} f(a) + b\varphi(b) - f(\varphi(b)) < ab, \text{ which is (8).}
\]

3. Applications

If we set \( F(x) = \int_0^x f(t)dt \), then from (6), (7) and (8), applied to \( F(x) \), follow the inequalities:

\[
ab \leq \int_0^a f(x)dx + \int_0^b \varphi(y)dy;
\]

\[
\int_0^a f(x)dx + \int_0^b \varphi(y)dy < f(a)a + b\varphi(b) - f(\varphi(b));
\]

\[
\int_0^a f(x)dx + \frac{a}{\varphi(b)} \int_0^b \varphi(y)dy < ab, \text{ if } a < \varphi(b);
\]

\[
\frac{b}{f(a)} \int_0^a f(x)dx + \int_0^b \varphi(y)dy < ab, \text{ if } a > \varphi(b).
\]

In (9), (10) and (11) \( \varphi(y) \) is the inverse function of \( f(x) \).

Equalities we have if and only if \( a = \varphi(b) \).
Evidently (9) is the classic Young’s inequality.

The equalities (10) and (11) are proved in [5].

It is an advantage of the inequalities (6), (7) and (8) over (9), (10) and (11) that when deriving them and applying them manifest form integration is not used which allows the inequalities (6), (7) and (8) to be applied in proving of inequalities in the secondary school.

As application of (6), we shall prove the inequalities (2), (4) and (5):

1. Let \( f(x) = \frac{x^p}{p} \), \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
y = f'(x) = x^{p-1} \Rightarrow \varphi(y) = y^{1/(p-1)} \quad \text{and from (6) follows}
\]

\[
ab \leq \frac{a^p}{p} + b \cdot \frac{1}{b^{p-1}} - \left( \frac{b^{1/(p-1)}}{p} \right)^p = \frac{a^p}{p} + b^{p-1} - b^{p-1} \cdot \frac{1}{p} \quad \Leftrightarrow
\]

\[
ab \leq \frac{a^p}{p} + b^{p-1} \left( 1 - \frac{1}{p} \right).
\]

Substituting \( 1 - \frac{1}{p} = \frac{1}{q} \), \( \frac{p}{p-1} = q \), we get (2).

Equality we have if \( a = b^{1/(p-1)} \).

2. Let \( f(x) = x^b \), \( b > 1 \). Then

\[
y = f'(x) = bx^{b-1} \Rightarrow \varphi(y) = \left( \frac{y}{b} \right)^{1/b-1} \quad \text{and from (6) follows}
\]

\[
ab \leq a^b + b \cdot 1 - 1, \quad \text{which is the inequality (4)}.
\]

Equality we have if \( a = 1 \).

3. Let \( f(x) = e^x - x \). Then

\[
y = f'(x) = e^x - 1 \Rightarrow \varphi(y) = \ln(1 + y) \quad \text{and from (6) follows}
\]

\[
ab \leq e^a - a + b \cdot \ln(1 + b) - \left( e^{\ln(1+b)} - \ln(1 + b) \right) =
\]

\[
e^a - a + \ln(1 + b)^b - (1 + b - \ln(1 + b)) \quad \Leftrightarrow
\]

\[
ab \leq e^a - a - b - 1 + \ln(1 + b)^{b+1}, \quad \text{which is the inequality (5)}.
\]

Equality we have if \( a = \ln(1 + b) \).

By (7) and (8) can be found the converse inequalities of (2), (3), (4) and (5).
4. Remarks

In conclusion we propose one more inequality which also may be interpreted as converse to Young’s inequality.

Let us go back to the Lagrange’s formula for \( f(x) \) in the interval \([a, \varphi(b)]\):

\[
f(\varphi(b)) - f(a) = f'(c)(\varphi(b) - a), \text{ where } a < c < \varphi(b).
\]

Since \( f'(c) > f'(a) \), then \( f(\varphi(b)) - f(a) > f'(a)(\varphi(b) - a) \)

\[
(12) \quad f(a) + b\varphi(b) - f(\varphi(b)) < f'(a)a + b\varphi(b) - f'(a)\varphi(b)
\]

The same inequality follows when \( a > \varphi(b) \).

If we apply (12) for \( F(x) = \int_a^x f(t)dt \), then we easily get:

\[
\int_a^a f(x)dx + \int_a^b \varphi(y)dy < f(a)a + b\varphi(b) - f(a)\varphi(b).
\]

Equality we have if \( a = \varphi(b) \).

References


Department of Mathematics
Technical University at Plovdiv
25, Tsanko Dyustabanov Str.
4000 Plovdiv
BULGARIA

e-mail: peyyyo@mail.bg
e-mail: pete_blood62@yahoo.com
An improperly posed problem for the one-dimensional heat equation

Valentina Proytcheva

Abstract
The improperly posed problem for the temperature \( u(x,t) \), \( x \in (0,1) \), \( t > 0 \) in a beam with prescribed values of \( u(x,0) \), \( x \in (0,1) \), \( u(0,t) \) and \( u_x(0,t) \), \( t > 0 \), but with unknown values of \( u(1,t) \), \( t > 0 \), is investigated.

1. Introduction
Let \( u(x,t) \), \( x \in \Omega \), \( t > 0 \) be the temperature distribution in a body \( \Omega \subset \mathbb{R}^3 \) at time \( t > 0 \). If \( u(x,t) \) is prescribed at time \( t = 0 \) for all \( x \) in \( \Omega \), as well as on the boundary \( \partial \Omega \) for all \( t > 0 \), then \( u(x,t) \) is the unique solution of the following initial boundary value problem

\[
\begin{cases}
\Delta u = u_t, & x \in \Omega, \ t > 0, \\
u(x,t) = g(x,t), & x \in \partial \Omega, \ t > 0 \\
u(x,0) = f(x), & x \in \Omega,
\end{cases}
\] (1.1)

and \( u(x,t) \) depends continuously on the data \( f \) and \( g \) of problem (1.1). Such a problem is said to be well-posed. Clearly \( u(x,t) \) fails to be unique if \( g \) is given only on a portion \( \Gamma \) of \( \partial \Omega \). In such a case one might compensate the lack of boundary data by prescribing not only \( u(x,t) \), but for instance \( \frac{\partial u}{\partial n} \) on \( \Gamma \), where \( \frac{\partial u}{\partial n} \) is the outward normal derivative of \( u \). However the resulting initial boundary value problem is not any more well-posed in the sense that a solution may fail to exist, and if it exists, it may not be unique, nor depend continuously on the data.
We refer the reader to the books of Payne [2] and of Ames and Straughan [1] dedicated to the extensive works on such improperly posed problems.

This note deals with the following improperly posed problem for the one-dimensional heat equation

$$\begin{cases} u_{xx}(x, t) = u_t(x, t), & x \in (0, 1), \quad t > 0, \\ u(0, t) = g(t), & t > 0, \\ u_x(0, t) = h(t), & t > 0, \\ u(x, 0) = f(x), & x \in (0, 1) \end{cases} \tag{1.2}$$

In (1.2), \( u(x, t) \) may be interpreted as the temperature distribution in a beam. No data are given at the extremity \( x = 1 \), but this lack of information is compensated by imposing not only the values of \( u \) but the values of \( u_x \) at \( x = 0 \) for all \( t > 0 \).

Following an argument of Payne and Philippin in [3], we introduce the auxiliary function \( v(x, t) \) defined as

$$v(x, t) := u(x, t) - xg(t), \quad x \in (0, 1), \quad t > 0, \tag{1.3}$$

which satisfies

$$\begin{cases} v_{xx} - v_t = xg'(t), & x \in (0, 1), \quad t > 0, \\ v(1, t) = 0, & t > 0, \\ v_x(1, t) = h(t) - g(t) = H(t), & t > 0, \\ v(x, 0) = f(x) - xg(0) =: F(x), & x \in (0, 1). \end{cases} \tag{1.4}$$

We now replace the non-well-posed problem (1.4) by the following modified problem

$$\begin{cases} w_{xx} - w_t = xg'(t), & x \in (0, 1), \quad t > 0, \\ w(1, t) + \alpha w(0, t) = 0, & t > 0, \\ w_x(1, t) + \beta w_x(0, t) = H(t), & t > 0, \\ w(x, 0) = F(x), & x \in (0, 1). \end{cases} \tag{1.5}$$

where \( \alpha \) and \( \beta \) are two nonnegative parameters.

We note that (1.5) reduces to (1.4) when \( \alpha = \beta = 0 \).

An approximation of \( u(x, t) \) will obviously be given by the function \( \tilde{u}(x, t) \) defined as

$$\tilde{u}(x, t) := w(x, t) + xg(t) \tag{1.6}$$
An improperly posed problem for the one-dimensional ... for small positive values of the parameters $\alpha, \beta$.

In the next section, we derive conditions on $\alpha, \beta$, sufficient to insure that problem (1.5) will be well-posed, so that $w(x, t)$ will depend continuously on the data $f, g, h$ under these conditions, implying at the same time the continuous dependence of $\tilde{u}(x, t)$.

2. A continuous dependence result

In this section we derive a first order differential inequality for the following quantity

$$\|w\|^2 := \int_0^1 (1 + \gamma x^2) w^2 dx,$$  \hfill (2.1)

valid under appropriate restrictions for $\alpha$ and $\beta$.

In (2.1), $\gamma$ is some nonnegative constant to be chosen later.

Differentiating (2.1), we obtain

$$\frac{d}{dt}(\|w\|^2) = 2 \int_0^1 [1 + \gamma x^2] w w_t dx$$

$$= 2 \int_0^1 [1 + \gamma x^2] (w_{xx} - x g'(t)) w dx$$

$$= 2 (1 + \gamma x^2) w w_x \bigg|_0^1 - 2 \int_0^1 (1 + \gamma x^2) w_x^2 dx$$

$$+ 2 \gamma \left[ \int_0^1 w^2 dx - w^2 (1, t) \right] - 2 g'(t) \int_0^1 (1 + \gamma x^2) w dx .$$

Making use of the boundary conditions in (1.5), we have

$$\left(1 + \gamma x^2\right) w w_x \bigg|_0^1 = -(1 + \gamma) \alpha w(0, t) [H(t) - \beta w_x(0, t)] -$$

$$- w(0, t) w_x(0, t).$$  \hfill (2.3)

Combining (2.2) and (2.3), we obtain

$$\frac{d}{dt}(\|w\|^2) = 2 \left[ (1 + \gamma) \alpha \beta - 1 \right] w(0, t) w_x(0, t)$$

$$- 2 \alpha (1 + \gamma) w(0, t) H(t) - 2 \int_0^1 (1 + \gamma x^2) w_x^2 dx$$

$$- 2 \gamma w^2(1, t) + 2 \gamma \int_0^1 w^2 dx - 2 g'(t) \int_0^1 (1 + \gamma x^2) x w dx .$$  \hfill (2.4)

Making use of the arithmetic-geometric-mean inequality, we have
\[-2 \alpha (1 + \gamma) w(0, t) H(t) \leq \frac{(1 + \gamma)^2 \alpha^2 H^2(t)}{\sigma} + \sigma w^2(0, t) \quad (2.5)\]

for arbitrary \( \sigma > 0 \), and

\[-2 g'(t) \int_0^1 (1 + \gamma x^2) xw dx \leq \frac{g'^2(t)}{\nu} \int_0^1 (1 + \gamma x^2)^2 x^2 dx + \nu \int_0^1 w^2 dx \quad (2.6)\]

for arbitrary \( \nu > 0 \).

Combining (2.4), (2.5), (2.6), neglecting a nonnegative term in (2.4), and selecting
\n\[
\gamma := \frac{1}{\alpha_\beta} - 1 > 0 \quad \text{for} \quad \alpha_\beta < 1, \quad (2.7)
\]

we obtain the inequality

\[
\frac{d}{dt}(\|w\|^2) \leq (2 \gamma + \nu) \int_0^1 w^2 dx + \left( \sigma - 2 \gamma \alpha^2 \right) w^2(0, t) - 2 \int_0^1 w_x^2 dx + \frac{(1 + \gamma)^2 \alpha^2 H^2(t)}{\sigma} + \frac{g'^2(t)}{\nu} \int_0^1 (1 + \gamma x^2)^2 x^2 dx. \quad (2.8)
\]

We obviously need a lower bound for the quantity \( \int_0^1 w^2 dx \).

Integrating by parts, we have for arbitrary \( \mu = const. \)

\[
2 \int_0^1 (1 + \mu x) w w_x dx = (1 + \mu) w^2 \bigg|_0^1 - \mu \int_0^1 w^2 dx = w^2(0, t)[1 + \mu - 1] - \mu \int_0^1 w^2 dx, \quad (2.9)
\]

from which we obtain for arbitrary \( \mu = const. > 0 \)

\[
\mu \int_0^1 w^2 dx + w^2(0, t)[1 - (1 + \mu) \alpha^2] = -2 \int_0^1 (1 + \mu) w w_x dx \quad (2.10)
\]

\[
\leq \frac{2}{\mu} \int_0^1 (1 + \mu x)^2 w_x^2 dx + \frac{\mu}{2} \int_0^1 w^2 dx.
\]

Solving for \( \int_0^1 w_x^2 dx \) leads to the inequality

\[
\int_0^1 w_x^2 dx \geq \frac{\mu^2}{4 (1 + \mu)^2} \int_0^1 w^2 dx + \frac{\mu [1 - (1 + \mu) \alpha^2]}{2 (1 + \mu)^2} w^2(0, t). \quad (2.11)
\]
Combining (2.8) and (2.11), we obtain the inequality

\[
\frac{d}{dt}(\|w\|^2) \leq -Lw^2(0,t) - K \int_0^1 w^2 dx + z(t),
\]

with

\[
L := \alpha^2 \left[ 2\gamma - \frac{\mu}{(1 + \mu)} \right] + \frac{\mu}{(1 + \mu)^2} - \sigma, \tag{2.13}
\]

\[
K := \frac{\mu^2}{2(1 + \mu)^2} - 2\gamma - \nu, \tag{2.14}
\]

\[
z(t) := \frac{(1 + \gamma)^2 \alpha^2 H^2(t)}{\sigma} + g^2(t) \int_0^1 (1 + \gamma x^2)^2 x^2 dx, \tag{2.15}
\]

for arbitrary \(\sigma, \mu, \nu > 0\).

We now select

\[
0 < \sigma < \frac{\mu}{(1 + \mu)^2} \tag{2.16}
\]

so that we have \(L \geq 0\) for \(\alpha\) small enough and \(\alpha\beta < 1\), leading to the inequality

\[
\frac{d}{dt}(\|w\|^2) \leq -K \int_0^1 w^2 dx + z(t). \tag{2.17}
\]

With

\[
-K \int_0^1 w^2 dx \leq \begin{cases} 
-\frac{K}{1 + \gamma} \|w\|^2, & K \geq 0, \\
-\frac{K}{\|w\|^2}, & K \leq 0,
\end{cases} \tag{2.18}
\]

we are led to the desired first order differential inequality for \(\|w\|^2\):

\[
\frac{d}{dt}(\|w\|^2) + \frac{K}{1 + \gamma} \|w\|^2 \leq z(t) \quad \text{if} \quad K \geq 0, \quad \text{or} \tag{2.19}
\]

\[
\frac{d}{dt}(\|w\|^2) + K \|w\|^2 \leq z(t) \quad \text{if} \quad K \leq 0. \tag{2.20}
\]

Integrating (2.19), (2.20) from 0 to \(t\), we obtain the following upper bounds for \(\|w\|^2\):

\[
\|w\|^2 \leq \begin{cases} 
\|F\|^2 \exp \left( -\frac{K}{1 + \gamma} t \right) + \int_0^t z(\eta) \exp \left( -\frac{K}{1 + \gamma} (t - \eta) \right) d\eta, & K \geq 0, \\
\|F\|^2 \exp \left( -K t \right) + \int_0^t z(\eta) \exp \left( -K (t - \eta) \right) d\eta, & K \leq 0,
\end{cases} \tag{2.21}
\]
valid for $\alpha \beta < 1$ and $\alpha$ sufficiently small.

We note that $K > 0$ if

$$\frac{\mu^2}{4(1+\mu)^2} > \gamma := \frac{1}{\alpha \beta} - 1,$$

i.e. if

$$\frac{1}{\alpha \beta} \leq \frac{\mu^2}{4(1+\mu)^2} + 1,$$

in which case $\|w\|\|\|$ decreases exponentially to zero when $t \to \infty$.

Moreover we have by the triangle inequality

$$\|F\| = \|f(x) - xg(0)\| \leq \|f\| + \|g(0)\|\|\|$$

$$\leq \sqrt{1 + \gamma \left( \int_0^1 f^2(x) \leq + \frac{g(0)}{\sqrt{\gamma}} \right)}.$$  \hspace{1cm} (2.22)

It follows from (2.21), (2.22) that $\|w\| \to 0$ on $[0, \tau]$ for arbitrary $\tau > 0$ as

$$\int_0^1 f^2(x) \to 0, \quad g(t) \to 0, \quad g'(t) \to 0 \quad \text{and} \quad h(t) \to 0$$

This establishes the continuous dependence of the solution $w(x,t)$ of (1.5) on any time interval $[0, \tau]$ for positive $\alpha, \beta$ with $\alpha \beta < 1$ and $\alpha$ sufficiently small.

The continuous dependence of $w(x,t)$ implies the continuous dependence of $\tilde{w}(x,t)$ defined by (1.6) since we have

$$\|\tilde{w}(x,t)\| \leq \|w\| + \|xg(t)\|$$ \hspace{1cm} (2.23)

by the triangle inequality.

References


Department of Mathematics
Technical University at Plovdiv
25, Tsanko Dyustabanov Str.
4000 Plovdiv
BULGARIA

e-mail: vproicheva@abv.bg
The exact distribution of the ump test for diagonality of covariance matrices with missing elements

Evelina Veleva

Abstract
This paper consider the uniformly most powerful (UMP) test for diagonality of the covariance matrix of the multivariate normal distribution, when the empirical correlation matrix has missing elements in the same column. The exact distribution of the test statistic is derived. Additionally, a new property of the Wishart distribution is obtained.

1. Introduction

Recently, the case of missing (unidentified) elements in the covariance and correlation matrices has received considerable attention in the literature (see [1] – [5]).

In this paper we assume that there are \( k \) missing elements in the empirical correlation matrix, located in the same column.

Without loss of generality, let these are the elements \( r_{i,n}, i=1,...,k \) in the last column of the empirical correlation matrix \( R = (r_{i,j}) \).

We suppose that the \( n \times n \) empirical correlation matrix \( R \) is obtained from a sample of size \( m \) from an \( n \)-variate normal distribution \( \mathcal{N}_n(\mu, \Sigma) \), where \( \mu \) and \( \Sigma \) are unknown. Consider the hypothesis \( H_0 \) for diagonality of the covariance matrix ,
\[
H_0 : \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2), \quad \sigma_i^2, \text{ are unknown, } i = 1, \ldots, n,
\]
against the alternative \( H_1 \) no constraints on \( \Sigma \).

Under the above assumptions for missing elements in \( R \), the UMP (uniformly most powerful) test for checking \( H_0 \) against \( H_1 \) is derived in [8].

In this paper we get the exact distribution of the test under \( H_0 \). Additionally, a new property of the Wishart distribution is obtained.

2000 Mathematics Subject Classification: 62H10.
Key words and phrases: covariance matrix, missing elements, product of beta random variables.
Received June 15, 2009.
2. Preliminary notes

Let $A$ be a real square matrix of order $n$. Let $\alpha$ and $\beta$ be nonempty subsets of the set $\mathbb{N}_n = \{1, \ldots, n\}$. By $A[\alpha, \beta]$ we denote the submatrix of $A$, composed of the rows with numbers from $\alpha$ and the columns with numbers from $\beta$. When $\beta \equiv \alpha$, $A[\alpha, \alpha]$ is denoted simply by $A[\alpha]$. For the complement of $\alpha$ in $N_n$ we use the notation $\alpha^c$.

The likelihood ratio test for $H_0$ against $H_1$ is shown in [8] to be

$$
\frac{L_0^*}{L_1^*} = \frac{\left( \det R[\{1, \ldots, k\}^c] \det R[\{n\}^c] \right)^{\frac{m-1}{2}}}{\det R[\{1, \ldots, k, n\}^c]^{\frac{m-n-1}{2}}}.
$$

**Definition 1.** A random matrix $V = (\nu_{i,j})$ is said to have distribution $\psi_n(m)$ with parameters $n, m, n < m$, written as $V \sim \psi_n(m)$, if $V$ is a symmetric matrix of order $n$ with units on the main diagonal and the joint density function of the elements above the main diagonal has the form

$$
f(y_{i,j}, 1 \leq i < j \leq n) = \left[ \Gamma\left(\frac{m}{2}\right) \right]^n \left[ \Gamma_n\left(\frac{m}{2}\right) \right]^{-1} (\det Y)^{-\frac{m-n-1}{2}},
$$

if $Y$ is a positive definite matrix, where $Y = (y_{i,j})$ is a symmetric matrix with units on the main diagonal.

By $\Gamma_n(\alpha)$ in the above definition is denoted the multivariate Gamma function,

$$
\Gamma_n(\alpha) = \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) \cdots \Gamma\left(\alpha - \frac{n-1}{2}\right).
$$

It is shown in [8], that under $H_0$ $R \sim \psi_n(m-1)$.

Let $\alpha \subset \mathbb{N}_n = \{1, \ldots, n\}$, $i, j \in \mathbb{N}_n$ and $i, j \not\in \alpha$. Suppose that in the submatrix $A[\alpha \cup \{i\}, \alpha \cup \{j\}]$ of the matrix $A = (a_{i,j})$ we replace the element $a_{i,j}$ with 0. We shall denote the obtained matrix by $A[\alpha \cup \{i\}, \alpha \cup \{j\}]^0$. The next Proposition can be found in [6].

**Proposition 1.** Let $A = (a_{i,j})$ be a real symmetric matrix of size $n$, and let $i, j$ be fixed integers such that $1 \leq i < j \leq n$. The matrix $A$ is positive definite if and only if the matrices $A[\{i\}^c]$ and $A[\{j\}^c]$ are positive definite and the element $a_{i,j}$ satisfies the inequalities...
The exact distribution of the ump test for diagonality of ...

\[
\frac{(-1)^j \det A \{ i \}^c, \{ j \}^c \}^0 - \sqrt{\det A \{ i \}^c \det A \{ j \}^c}}{\det A \{ i, j \}^c} < a_{i,j}
\]

\[
< \frac{(-1)^j \det A \{ i \}^c, \{ j \}^c \}^0 + \sqrt{\det A \{ i \}^c \det A \{ j \}^c}}{\det A \{ i, j \}^c}.
\]

3. The exact distribution of the test under \( H_0 \)

**Theorem 1.** A real symmetric matrix \( A = (a_{i,j}) \) of size \( n \) is positive definite if and only if its elements satisfy the following conditions:

\[
a_{i,i} > 0, \quad i = 1, \ldots, n;
\]

\[
-a_{i,i} a_{i+1,i+1} < a_{i,i+1} < a_{i,i+1} a_{i+1,i+1}, \quad i = 1, \ldots, n - 1;
\]

\[
\left( (-1)^j \det A \{ i, \ldots, j - 1 \}, \{ i + 1, \ldots, j \} \right)^0 - \frac{\sqrt{\det A \{ i, \ldots, j - 1 \} \det A \{ i + 1, \ldots, j \}}}{\det A \{ i + 1, \ldots, j - 1 \}} \right) <
\]

\[
< \left( (-1)^j \det A \{ i, \ldots, j - 1 \}, \{ i + 1, \ldots, j \} \right)^0 + \frac{\sqrt{\det A \{ i, \ldots, j - 1 \} \det A \{ i + 1, \ldots, j \}}}{\det A \{ i + 1, \ldots, j - 1 \}} \right)
\]

\[
i = 1, \ldots, n - 2; \quad j = i + 2, \ldots, n.
\]

**Proof.** The proof is by induction on \( n \). On each step we apply Proposition 1. □

Let \( P( n, \mathbb{R} ) \) be the set of all real, symmetric, positive definite matrices of order \( n \) . Let us denote by \( D( n, \mathbb{R} ) \) the set of all real, symmetric matrices of order \( n \) , with positive diagonal elements, which off-diagonal elements are in the interval (-1,1). There exist a bijection (one-to-one correspondence) \( h : D( n, \mathbb{R} ) \rightarrow P( n, \mathbb{R} ) \), considered in [7].

**Theorem 2.** Let \( h \) is a function from \( D( n, \mathbb{R} ) \) to \( P( n, \mathbb{R} ) \), assigning to each matrix \( X = (x_{i,j}) \) from \( D( n, \mathbb{R} ) \) a symmetric matrix \( Y = (y_{i,j}) \) from \( P( n, \mathbb{R} ) \), such that

\[
y_{i,i} = x_{i,i}, \quad i = 1, \ldots, n;
\]

\[
y_{i,i+1} = x_{i,i+1} \sqrt{y_{i,i} y_{i+1,i+1}}, \quad i = 1, \ldots, n - 1;
\]

\[
y_{i+1,i} = x_{i+1,i} \sqrt{y_{i+1,i} y_{i,i}}, \quad i = 1, \ldots, n - 1;
\]

\[
y_{i+1,i+1} = x_{i+1,i+1} \sqrt{y_{i+1,i+1} y_{i,i+1}}.
\]
Evelina Veleva

\[ y_{i,j} = \]
\[ = \left( (-1)^{j-i} \det Y[\{i, \ldots, j-1\}, \{i+1, \ldots, j\}]^0 + \frac{x_{i,j} \sqrt{\det Y[\{i, \ldots, j-1\}] \det Y[\{i+1, \ldots, j\}]}}{\det Y[\{i+1, \ldots, j-1\}] \det Y[\{i+1, \ldots, j\}]} \right) \]
\[ i = 1, \ldots, n-2, \quad j = i + 2, \ldots, n \quad (4) \]

Then \( \tilde{h} \) is a bijection.

**Proof.** Since \(-1 < x_{i,j} < 1 \) for \( i \neq j \), using Theorem 1 it can be seen that the matrix \( Y = (y_{i,j}) \) is positive definite. Therefore all principal minors of \( Y \) under the square root in (4) are positive. The equalities (2) – (4) define uniquely the symmetric matrix \( Y \). The elements on the main diagonal are determined firstly. The elements on the next diagonal parallel to the previous one then follow. Proceeding this way, finally the element \( y_{1,n} \) is defined. Let \( Y = (y_{i,j}) \) be an arbitrary matrix from \( P(\mathbb{R}) \). Let us consider the matrix \( X = (x_{i,j}) \), such that

\[ x_{i,i} = y_{i,i} , \quad i = 1, \ldots, n ; \quad (5) \]

\[ x_{i,i+1} = x_{i+1,i} = \frac{y_{i,i+1}}{\sqrt{y_{i,i}y_{i+1,i+1}}} , \quad i = 1, \ldots, n-1 ; \quad (6) \]

\[ x_{i,j} = x_{j,i} = \]
\[ = y_{i,j} \det Y[\{i+1, \ldots, j-1\}] + (-1)^{j-i-1} \det Y[\{i, \ldots, j-1\}, \{i+1, \ldots, j\}]^0 \]
\[ \sqrt{\det Y[\{i, \ldots, j-1\}] \det Y[\{i+1, \ldots, j\}]} \]
\[ \]
\[ = \frac{(-1)^{j-i-1} \det Y[\{i, \ldots, j-1\}, \{i+1, \ldots, j\}]}{\sqrt{\det Y[\{i, \ldots, j-1\}] \det Y[\{i+1, \ldots, j\}]}} , \]
\[ i = 1, \ldots, n-2, \quad j = i + 2, \ldots, n . \quad (7) \]

Since \( Y \) is a positive definite matrix, using Theorem 1 it can be checked that the matrix \( X = (x_{i,j}) \) is from \( D(n, \mathbb{R}) \). By formulas (2) – (4) it can be seen that \( \tilde{h}(X) = Y \). If \( X_1 \neq X_2 , \quad X_1, X_2 \in D(n, \mathbb{R}) \) then from equalities (2) – (4) it follows that \( \tilde{h}(X_1) \neq \tilde{h}(X_2) \). □

**Definition 2.** A random variable \( X \) is said to have a beta distribution, written as \( X \sim Beta(\alpha, \beta, a, b) \), if its probability density function is given by

\[ f(x) = \frac{\Gamma(\alpha + \beta) (x - a)^{\alpha-1} (b - x)^{\beta-1}}{\Gamma(\alpha) \Gamma(\beta) (b - a)^{\alpha+\beta-1}} , \quad a < x < b . \]
Theorem 3. Let $\xi=(\xi_{i,j})$ be a symmetric $n\times n$ random matrix with units on the main diagonal. Let $\xi_{i,j} \sim \text{Beta}\left((m-j+i)/2,(m-j+i)/2,-1,1\right)$, $1 \leq i < j \leq n$ be mutually independent; $m$ is an integer, $m > n$. Then the matrix $\eta = \tilde{h}(\xi) \sim \psi_n(m)$.

Proof. The joint density function of the random variables $\xi_{i,j}, 1 \leq i < j \leq n$ has the form

$$f(x_{i,j}, 1 \leq i < j \leq n) = \left[ \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right) \right]^{-1} \prod_{1 \leq i < j \leq n} \left(1 - x_{i,j}^2 \right)^\frac{m-j+i-2}{2} ,$$

where $x_{i,j} \in (-1,1)$, $1 \leq i < j \leq n$.

The equalities (5) – (7) give the inverse transformation formulas. Here $x_{i,i} = y_{i,i} = 1$, $i = 1, \ldots, n$, since $\xi_{i,i} = 1$, $i = 1, \ldots, n$. Using the properties of determinants it can be proved the next statement.

Proposition 2. Let $A$ be a real square matrix of order $n$. Let $i$, $j$ be integers, $1 \leq i < j \leq n$. Then

$$\det A \det A^{\{i,j\}} = \det A^{\{i\}} \det A^{\{j\}} - \det A^{\{i\}} \det A^{\{j\}} \det A^{\{i,j\}} .$$

Applying Proposition 2 on the matrix $Y^{\{i,\ldots,j\}}$, it follows from (7) that

$$1 - x_{i,j}^2 = \frac{\det Y^{\{i,\ldots,j\}} \det Y^{\{i+1,\ldots,j-1\}}}{\det Y^{\{i,\ldots,j-1\}} \det Y^{\{i+1,\ldots,j\}}} , 2 \leq i + 1 < j \leq n .$$

The joint density function (8) now can be expressed by the variables $y_{i,j}, 1 \leq i < j \leq n$:

$$f(y_{i,j}, 1 \leq i < j \leq n) =$$

$$= \left[ \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right) \right]^{-1} \left( \det Y \right)^\frac{m-n-1}{2} \prod_{j=2}^{n-1} \sqrt{\det Y^{\{1,\ldots,j\}}} \det Y^{\{j,\ldots,n\}} .$$

According to Theorem 1, the conditions (9) are equivalent to the positive definiteness of the symmetric matrix $Y = (y_{i,j})$. Finally, we find the Jacobian $J$ of the transformation and calculate its determinant

$$\det J = \prod_{j=2}^{n-1} \frac{1}{\sqrt{\det Y^{\{1,\ldots,j\}}} \det Y^{\{j,\ldots,n\}}} .$$
Theorem 4. Let $V \sim \psi_n(m)$. Then for arbitrary integers $p$ and $q$, \(1 < p \leq q \leq n\) the random variable $\frac{\det V[\{1, \ldots, q\}] \det V[\{p, \ldots, n\}]}{\det V[\{p, \ldots, q\}]}$ is distributed as the product $\zeta_1 \ldots \zeta_{n-1}$, where $\zeta_1, \ldots, \zeta_{n-1}$ are independent random variables, $\zeta_r \sim \text{Beta}\left((m - q + r)/2, (q - r)/2, 0, 1\right)$, $r = 1, \ldots, p - 1$, $\zeta_r \sim \text{Beta}\left((m - n + r)/2, (n - r)/2, 0, 1\right)$, $r = p, \ldots, n - 1$.

Proof. From Theorem 3 it follows that the matrix $V$ can be considered as an image $V = \tilde{h}(\xi)$ of a symmetric matrix $\xi = (\xi_{i,j})$, where $\xi_{i,j}, 1 \leq i < j \leq n$ are independent random variables with Beta distribution and $\xi_{i,i} = 1, i = 1, \ldots, n$. Using (10), it can be shown by induction on $j$, that for every integers $i$ and $j$, $1 \leq i < j \leq n$

$$\det V[\{i, \ldots, j\}] = (1 - \xi^2_{i,i+1})(1 - \xi^2_{i,i+2}) \ldots (1 - \xi^2_{i,j}) \det V[\{i+1, \ldots, j\}]. \tag{11}$$

Using (11) we obtain the equality

$$\det V[\{i, \ldots, j\}] = \prod_{r=i}^{j-1} \prod_{s=r+1}^{j} (1 - \xi^2_{r,s}).$$

Hence, for arbitrary integers $p$ and $q$, $1 < p \leq q \leq n$ we get the representation

$$\frac{\det V[\{1, \ldots, q\}] \det V[\{p, \ldots, n\}]}{\det V[\{p, \ldots, q\}]} = \left(\prod_{r=1}^{p-1} \prod_{s=r+1}^{q} (1 - \xi^2_{r,s})\right) \left(\prod_{r=p}^{n-1} \prod_{s=r+1}^{n} (1 - \xi^2_{r,s})\right). \tag{12}$$

Let us denote by $\zeta_1, \ldots, \zeta_{n-1}$ the random variables

$$\zeta_r = \prod_{s=r+1}^{q} (1 - \xi^2_{r,s}), \quad r = 1, \ldots, p - 1, \quad \zeta_r = \prod_{s=r+1}^{n} (1 - \xi^2_{r,s}), \quad r = p, \ldots, n - 1.$$

They are independent because of the independence of $\xi_{i,j}, 1 \leq i < j \leq n$. It can be proved that

$$1 - \xi^2_{i,j} \sim \text{Beta}\left((m - j + i)/2, 1/2, 0, 1\right), 1 \leq i < j \leq n.$$

Now we use the known property “The product of independent $\text{Beta}(\alpha, \gamma, 0, 1)$ and $\text{Beta}(\alpha + \gamma, \delta, 0, 1)$ is $\text{Beta}(\alpha, \gamma + \delta, 0, 1)$ distributed” to complete the proof. □


Department of Numerical Methods
and Statistics
Rousse University “A. Kanchev”
8, Studentska Str.
7017 Rousse
BULGARIA
E-mail: eveleva@ru.acad.bg
Uniformly distributed positive definite matrices with bounded trace

Evelina Veleva

Abstract

This paper consider the space of all real positive definite matrices of order $n$, with trace (the sum of all eigenvalues) in a given interval $(a, b]$, $0 \leq a < b$. The volume of this space is derived and the uniform distribution over it is defined. The paper gives an algorithm for generation of uniformly distributed random matrices over this space. Some interesting properties are obtained.

Key words: random matrix theory, eigenvalues, bounded trace, uniform distribution, positive definite matrix.

1. Introduction

A sufficient condition for applying many numerical algorithms is the positive definiteness of a matrix. The diagonal elements and the eigenvalues of the matrix have often special significance. The correctness of such numerical algorithm can be proven if we are able to choose a positive definite matrix uniformly at random.

Let $P(n, \mathbb{R})$ be the space of all real, symmetric, positive definite matrices of order $n$. The whole set $P(n, \mathbb{R})$ is a cone with infinite volume. This enforces to introduce additional restrictions on the space $P(n, \mathbb{R})$, which to reduce our choice within a set with finite volume.

Let us denote by $P(n, \mathbb{R}; c_1, \ldots, c_n)$ the set of all elements of $P(n, \mathbb{R})$, whose diagonal elements are equal to the positive constants $c_1, \ldots, c_n$. An algorithm for generation of uniform distributed random matrices over $P(n, \mathbb{R}; c_1, \ldots, c_n)$ is proposed in [5].

This paper consider the set $P^{(a,b)}(n, \mathbb{R})$ of all elements of $P(n, \mathbb{R})$ with trace (the sum of all eigenvalues) in a given interval $(a, b]$, $0 \leq a < b$.

An algorithm for generation of uniform distributed random matrices over $P^{(a,b)}(n, \mathbb{R})$ is derived. Some interesting properties are also obtained.

2000 Mathematics Subject Classification: 62H10.

Key words and phrases: covariance matrix, missing elements, product of beta random variables.

Received June 15, 2009.
2. Preliminary notes

Let $D(n, \mathbb{R})$ be the set of all real, symmetric matrices of order $n$, with positive diagonal elements, which off-diagonal elements are in the interval (-1,1).

There exist a bijection (one-to-one correspondence) $h : D(n, \mathbb{R}) \rightarrow P(n, \mathbb{R})$, considered in [5], [6] and [8]. The image of an arbitrary matrix $X = (x_{i,j})$ from $D(n, \mathbb{R})$ by the bijection $h$, is a matrix $Y = (y_{i,j})$ from $P(n, \mathbb{R})$, such that

\[
y_{j,j} = x_{j,j}, \quad j = 1, \ldots, n,  \tag{1}
y_{1,j} = x_{1,j} \sqrt{x_{1,1} x_{j,j}}, \quad j = 2, \ldots, n,  \tag{2}
\]

\[
y_{i,j} = \sqrt{x_{i,i} x_{j,j}} \left[ \sum_{r=1}^{i-1} x_{r,i} x_{r,j} \prod_{q=1}^{r-1} \sqrt{1-x_{q,i}^2} (1-x_{q,j}^2) \right]
\quad + x_{i,j} \prod_{q=1}^{i-1} \sqrt{1-x_{q,i}^2} (1-x_{q,j}^2), \quad 2 \leq i < j \leq n.  \tag{3}
\]

**Definition 1.** A random matrix $\eta = (\eta_{i,j})$ is said to have a uniform distribution $U(P(n, \mathbb{R}; c_1, \ldots, c_n))$ over the set $P(n, \mathbb{R}; c_1, \ldots, c_n)$, where $n$ is an integer and $c_1, \ldots, c_n$ are positive constants, if its density function has the form

\[
f_{\eta}(Y) = \left[ \frac{1}{\Gamma \left( \frac{n+1}{2} \right)} \right]^n \left[ \frac{1}{\Gamma_n \left( \frac{n+1}{2} \right)} \right]^{-1} (c_1 \ldots c_n)^{-(n-1)/2}, \quad Y \in P(n, \mathbb{R}; c_1, \ldots, c_n).
\]

By $\Gamma_n(\alpha)$ in the above definition is denoted the multivariate Gamma function,

\[
\Gamma_n(\alpha) = \pi^{\frac{n(n-1)}{4}} \Gamma(\alpha) \Gamma \left( \alpha - \frac{1}{2} \right) \ldots \Gamma \left( \alpha - \frac{n-1}{2} \right).
\]

**Definition 2.** A random variable $\xi$ is said to have a beta distribution, written as $\xi \sim \text{Beta}(\alpha, \beta, a, b)$, if its density function is given by

\[
f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{(x-a)^{\alpha-1}(b-x)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}, \quad a < x < b.
\]

The next statement can be found in [5].
Proposition 1. Let \( \xi = (\xi_{i,j}) \) be a symmetric \( n \times n \) random matrix with diagonal elements equal to given positive constants \( c_1, \ldots, c_n \) respectively.

Let \( \xi_{i,j}, 1 \leq i < j \leq n \) be mutually independent and

\[
\xi_{i,j} \sim \text{Beta}\left((n - i + 1)/2, (n - i + 1)/2, -1, 1\right).
\]

Then the matrix \( \eta = h(\xi) \) is distributed \( U( P( n, \mathbb{R}; c_1, \ldots, c_n)) \).

From the definition of the bijection \( h \) by formulas (1) – (3) it follows the next Proposition.

Proposition 2. Let \( X = (x_{i,j}) \in \mathcal{D}(n, \mathbb{R}) \). Let us replace all the diagonal elements of \( X \) with 1 and denote the obtained matrix by \( X_1 \). Then \( h(X) = A h(X_1) A^T \), where \( A \) is the diagonal matrix \( A = \text{diag}\left(\sqrt{c_1}, \ldots, \sqrt{c_n}\right) \).

From Proposition 1 and 2 it follows that if \( \eta \sim U( P( n, \mathbb{R}; 1, \ldots, 1)) \) and \( c_1, \ldots, c_n \) are positive constants then the matrix

\[
\text{diag}\left(\sqrt{c_1}, \ldots, \sqrt{c_n}\right) \eta \text{diag}\left(\sqrt{c_1}, \ldots, \sqrt{c_n}\right)
\]

is uniform distributed \( U( P( n, \mathbb{R}; c_1, \ldots, c_n)) \).

Definition 3. A random matrix \( \eta \) is said to have distribution \( \psi_n(m) \) with parameters \( n, m, n < m \), if its density function has the form

\[
f_\eta(Y) = \left[\Gamma\left(\frac{m}{2}\right)\right]^n \left[\Gamma_n\left(\frac{m}{2}\right)\right]^{-1} \left(\det Y\right)^{\frac{m-n-1}{2}}, \quad Y \in P( n, \mathbb{R}; 1, \ldots, 1) .
\]

Let \( R \) be the empirical correlation matrix, obtained from a sample of size \( m \) from an \( n \)-variate normal distribution \( \mathcal{N}_n(\mu, \Sigma) \), where \( \mu \) and \( \Sigma \) are unknown, and \( \Sigma \) is a diagonal matrix. Then the distribution of \( R \) is \( \psi_n(m) \). The distribution \( U( P( n, \mathbb{R}; 1, \ldots, 1)) \) is a special case of \( \psi_n(m) \), when \( m = n + 1 \).

Properties and marginal densities of \( \psi_n(m) \) are considered in [2] – [8].

3. Main results

Definition 4. Random variables \( \xi_1, \ldots, \xi_n \) is said to have joint multivariate Liouville distribution \( L_n[ g(\cdot); \alpha_1, \ldots, \alpha_n] \), if its joint density function is proportional to

\[
x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} g(x_1 + \cdots + x_n),
\]

where the variables range over the orthant \( \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i > 0, \ i = 1, \ldots, n\} \), \( g \) is a continuous function and \( \alpha_1, \ldots, \alpha_n \) are positive numbers. The support of \( g \) can be either the interval \((0, \infty)\), or a finite interval in it (see [1]).
Let in the above definition the function $g$ be the indicator function $1_{(a,b]}$ of a given interval $(a,b]$, $0 \leq a < b$.

**Lemma 1.** Let $\xi_1, \ldots, \xi_n$ be random variables with joint distribution $L_n[1_{(a,b]}; \alpha_1, \ldots, \alpha_n]$. Then their joint density function is given by

$$f(x_1, \ldots, x_n) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} \left( b^{\alpha_1 + \cdots + \alpha_n} - a^{\alpha_1 + \cdots + \alpha_n} \right) x_1^\alpha_1 \cdots x_n^\alpha_n - 1,$$

where the variables range over the domain

$$d(\alpha_1, \ldots, \alpha_n) = \{(x_1, \ldots, x_n) \mid x_i > 0, i = 1, \ldots, n, \sum_{i=1}^n x_i \in (a,b]\}.$$

**Proof.** For $a = 0$ and $b = 1$ (4) gives the density function of the multivariate Dirichlet distribution $D(\alpha_1, \ldots, \alpha_n; 1)$ (see [1]).

Therefore we have

$$\int_{d(0,1]} x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} dx_1 \cdots dx_n = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)}.$$

If we substitute in (5) $x_i = y_i/c$, $i = 1, \ldots, n$, where $c$ is a constant, we get

$$\int_{d(0,c]} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1} dy_1 \cdots dy_n = c^{\alpha_1 + \cdots + \alpha_n} \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)}.$$

Since $d(0,a] \cup d(a,b] = d(0,b]$, we obtain

$$\int_{d(a,b]} y_1^{\alpha_1 - 1} \cdots y_n^{\alpha_n - 1} dy_1 \cdots dy_n = \int_{d(0,b]} - \int_{d(0,a]} = (b^{\alpha_1 + \cdots + \alpha_n} - a^{\alpha_1 + \cdots + \alpha_n}) \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)}.$$

**Definition 5.** A random matrix $\eta = (\eta_{i,j})$ is said to have uniform distribution $U(P(a,b)(n,\mathbb{R}))$ over the set $P(a,b)(n,\mathbb{R})$, where $n$ is an integer and $a, b$ are real numbers, $0 \leq a < b$, if its density function is given by

$$f(\eta(Y)) = K_n \cdot Y \in P(a,b)(n,\mathbb{R}),$$

where

$$K_n = \Gamma\left(\frac{n(n+1)}{2} + 1\right) \left[ \Gamma_n\left(\frac{n+1}{2}\right) \right]^{-1} \left[ \frac{n(n+1)}{2} - \frac{n(n+1)}{2} \right]^{-1}.$$
Theorem 1. Let $\xi=(\xi_{i,j})$ be a symmetric $n \times n$ random matrix.

Suppose that $\xi_{i,j}, \ 1 \leq i < j \leq n$ are mutually independent and $\xi_{i,j} \sim \text{Beta}\left(\frac{n-i+1}{2}, \frac{n-i+1}{2}, -1, 1\right)$. Let $\xi_{i,i}, \ i = 1, \ldots, n$ be independent of $\xi_{i,j}, \ 1 \leq i < j \leq n$ random variables with joint distribution $L_n \left[1_{(a,b)}; (n+1)/2, \ldots, (n+1)/2\right]$, where $a, b$ are real numbers, $0 \leq a < b$.

Then the matrix $\eta = h(\xi)$ has uniform distribution $U( P(a,b)(n, \mathbb{R}))$.

Proof. The joint density function of $\xi_{i,j}, \ 1 \leq i \leq j \leq n$ is

$$f_{\xi_{i,j}}(x_{i,j}, 1 \leq i \leq j \leq n) = K_n(x_{1,1} \ldots x_{n,n})^{(n-1)/2} \prod_{1 \leq i < j \leq n} (1 - x_{i,j}^2)^{(n-i-1)/2},$$

$$x_{i,j} \in (-1,1), 1 \leq i < j \leq n, x_{i,i} > 0, i = 1, \ldots, n, \sum_{i=1}^{n} x_{i,i} \in (a, b].$$

The new variables are the elements $\eta_{i,j}, 1 \leq i \leq j \leq n$ of the matrix $\eta = h(\xi)$, where $h$ is the bijection, defined by (1) – (3). It is shown in [5], that

$$\prod_{1 \leq i < j \leq n} (1 - x_{i,j}^2)^{(n-i-1)/2} = (x_{1,1} \ldots x_{n,n})^{-(n-1)/2} |\det J|^{-1},$$

where $J$ is the Jacobian of the transformation. Since the matrix $X = (x_{i,j})$ is from $D(n, \mathbb{R})$, its corresponding matrix $Y = (y_{i,j}), Y = h(X)$ is from $P(n, \mathbb{R})$.

From (1) it follows that $\sum_{i=1}^{n} y_{i,i} \in (a, b].$ Consequently, the joint density function of $\eta_{i,j}, 1 \leq i \leq j \leq n$ is

$$f_{\eta_{i,j}}(y_{i,j}, 1 \leq i \leq j \leq n) = K_n, Y \in P(a,b)(n, \mathbb{R}). \quad \square$$

Let $\eta \sim U( P(n, \mathbb{R}; 1, \ldots, 1))$ and $\xi_1, \ldots, \xi_n$ be independent of $\eta$ random variables with joint distribution $L_n \left[1_{(a,b)}; (n+1)/2, \ldots, (n+1)/2\right]$, where $a, b$ are real numbers, $0 \leq a < b$.

From Theorem 1 and Proposition 1 and 2 it follows that the matrix $\text{diag}\left(\sqrt{\xi_1}, \ldots, \sqrt{\xi_n}\right) \eta \text{diag}\left(\sqrt{\xi_1}, \ldots, \sqrt{\xi_n}\right)$ has uniform distribution $U( P(a,b)(n, \mathbb{R})).$

For generation of Liouville distributed random variables the next stochastic representation, given in [1] can be used.

Proposition 3. If $(\xi_1, \ldots, \xi_n) \sim L_n \left[g(\cdot); \alpha_1, \ldots, \alpha_n\right]$ then

$$(\xi_1, \ldots, \xi_n) \overset{d}{=} (\tau_1, \ldots, \tau_n) \nu, \text{ where } \nu = \xi_1 + \cdots + \xi_n \sim L_1 \left[g(\cdot); \alpha\right],$$

$$\alpha = \alpha_1 + \cdots + \alpha_n$$

and $(\tau_1, \ldots, \tau_n)$ is independent of $\nu$. 
Furthermore \( \tau_n = 1 - \sum_{i=1}^{n-1} \tau_i \); and \((\tau_1, \ldots, \tau_{n-1})\) has Dirichlet distribution \(D(\alpha_1, \ldots, \alpha_{n-1}; \alpha_n)\).

According to Lemma 1, the distribution \(L_1[1_{(a,b)}; \alpha]\) has density function of the form

\[
f(x) = \alpha (b^\alpha - a^\alpha)^{-1} x^{\alpha-1}, \quad x \in (a,b].
\]

(6)

**Theorem 2.** Let \(\zeta\) is a random variable with uniform distribution \(U(0,1)\). Then for \(0 \leq a < b\) and \(\alpha > 0\) the variable \((a^\alpha + \zeta( b^\alpha - a^\alpha ))^{1/\alpha}\) has density function (6).

Proof. The distribution function, corresponding to the density function (6) is

\[
F(x) = \frac{\alpha}{b^\alpha - a^\alpha} \int_a^x t^{\alpha-1} dt = \frac{x^{\alpha} - a^\alpha}{b^\alpha - a^\alpha}, \quad x \in (a,b].
\]

Then for \(y \in (a,b]\) the inverse function \(F^{-1}(y)\) is

\[
F^{-1}(y) = (a^\alpha + y(b^\alpha - a^\alpha ))^{1/\alpha}.
\]

Consequently, if \(\zeta \sim U(0,1)\) then \(F^{-1}(\zeta) \sim L_1[1_{(a,b)}; \alpha]\). \(\square\)

The next property follows directly from Lemma 1, Theorem 1 and Proposition 3.

**Corollary 1.** Let \(\eta \sim U(P^{(a,b)}((n, \mathbb{R})))\). Then the trace of \(\eta\) is distributed \(L_1[1_{(a,b)}; \alpha]\), where \(\alpha = n(n+1)/2\).

An interesting property for Beta distributions follows from the next Proposition, which is proven in [2].

**Proposition 4.** Let \(\eta = (\eta_{i,j}) \sim \psi_n(m)\). Then \(\eta_{i,j}, 1 \leq i \leq j \leq n\) are identically distributed \(\text{Beta}\left((m-1)/2, (m-1)/2, -1, 1\right)\).

Since the distribution \(U(P^{(n, \mathbb{R}; 1, \ldots, 1)})\) is a special case of \(\psi_n(m)\) when \(m = n + 1\), from Proposition 1, 4 and the equality (3) we get the next Corollary.

**Corollary 2.** Let \(\tau_1, \ldots, \tau_{k-1}, \nu_1, \ldots, \nu_k\) be independent random variables, \(\tau_i, \nu_i \sim \text{Beta}\left((n-i+1)/2, (n-i+1)/2, -1, 1\right), \quad i = 1, \ldots, k\), where \(n\) is an integer, \(n > k\). Then the random variable

\[
\tau_1 \nu_1 + \tau_2 \nu_2 \sqrt{(1 - \tau_1^2)(1 - \nu_1^2)} + \tau_3 \nu_3 \sqrt{(1 - \tau_1^2)(1 - \nu_1^2)(1 - \tau_2^2)(1 - \nu_2^2)} + \cdots + \tau_{k-1} \nu_{k-1} \sqrt{(1 - \tau_1^2)(1 - \nu_1^2) \cdots (1 - \tau_{k-2}^2)(1 - \nu_{k-2}^2)} \] 

\[+ \nu_k \sqrt{(1 - \tau_1^2)(1 - \nu_1^2) \cdots (1 - \tau_{k-1}^2)(1 - \nu_{k-1}^2)}
\]
Uniformly distributed positive definite matrices with...... has distribution $Beta\left(\frac{n}{2}, \frac{n}{2}, -1, 1\right)$.

**Proof.** Let $\eta = (\eta_{i,j}) \sim U( P( n, \mathbb{R}; 1, \ldots , 1))$. According to Proposition 4, $\eta_{i,j}$, $1 \leq i \leq j \leq n$ are identically distributed $Beta\left(\frac{n}{2}, \frac{n}{2}, -1, 1\right)$. From Proposition 1 we have that $\eta$ can be considered as the image $\eta = h(\xi)$ of a random matrix $\xi$ with mutually independent elements $\xi_{i,j} \sim Beta\left(\frac{n-i+1}{2}, \frac{n-i+1}{2}, -1, 1\right)$, $1 \leq i \leq j \leq n$ and units on the main diagonal.

For $i = k$ by equality (3) we get

$$
\eta_{k,j} = \xi_{1,k} \xi_{1,j} + \xi_{2,k} \xi_{2,j} \sqrt{(1 - \xi_{2,1,k}^2)(1 - \xi_{2,1,j}^2)} + \\
\xi_{3,k} \xi_{3,j} \sqrt{(1 - \xi_{1,1,k}^2)(1 - \xi_{1,1,j}^2)(1 - \xi_{2,1,k}^2)(1 - \xi_{2,1,j}^2)} + \ldots \\
+ \xi_{k-1,k} \xi_{k-1,j} \sqrt{(1 - \xi_{1,1,k}^2)(1 - \xi_{1,1,j}^2)(1 - \xi_{2,1,k}^2)(1 - \xi_{2,1,j}^2)} + \ldots \\
+ \xi_{k,j} \sqrt{(1 - \xi_{1,1,k}^2)(1 - \xi_{1,1,j}^2)(1 - \xi_{2,1,k}^2)(1 - \xi_{2,1,j}^2)}.
$$

To complete the proof it remains to substitute $\tau_s = \xi_{s,k}$, $s = 1, \ldots , k - 1$, $\nu_s = \xi_{s,j}$, $s = 1, \ldots , k$.

**References**


Department of Numerical Methods and Statistics
Rousse University “A. Kanchev”
8, Studentska Str.
7017 Rousse
BULGARIA
E-mail: eveleva@ru.acad.bg
Linear integral inequalities involving “MAXIMA”

S. S. Gluhcheva, K. V. Stefanova

Abstract

The purpose of the present paper is to establish some new integral inequalities, which provide explicit bounds on unknown functions and generalize the well known Gronwall inequality. The maximum of the unknown function on a certain interval is also involved into the right part of the inequality. The effectiveness of the obtained explicit bound of the unknown function is illustrated on some applications of studying properties of solutions of differential equations with “maxima” such as uniqueness, continuous dependence of initial conditions.

1. Introduction

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity ([2], [5]). Such kind of problems could be adequately modeled by differential equations which contain the maxima operator. Note that equations, involving maxima of the unknown function are called differential equations with "maximum". A. D. Mishkis also points out the necessity to study differential equations with "maximum" in his survey [3]. Some qualitative properties of the solutions of ordinary differential equations with "maximum" have been obtained in ([1], [4]).

2. Main Results

We will prove an analog of the classical inequality of Gronwall-Bellman, when a maximum is involved in.

**Theorem 1:** Let following conditions be satisfied:

1) Functions \( g, r \in C([t_0, \infty), R_+) \) and function \( \varphi \in C([t_0 - h, t_0], R_+) \).

2) Function \( x \in C([t_0, \infty), R_+) \) and it satisfies the inequalities

2000 Mathematics Subject Classification: 34D20.

Key words and phrases: integral inequalities, continuous dependence.

Received June 15, 2009.
\[
x(t) \leq C + \int_{t_0}^{t} g(s)x(s)ds + \int_{t_0}^{t} r(s) \max_{\zeta \in [s-h,s]} x(\zeta)ds, \quad t \geq t_0,
\]
(1)

\[
x(s) = \varphi(s) \text{ for } s \in [t_0 - h, t_0], \text{ where } h = \text{const} \geq 0, \quad C \leq \varphi(t_0).
\]
(2)

Then for \( t \geq t_0 \) inequality

\[
x(t) \leq \max_{s \in [t_0 - h, t_0]} \varphi(s) e^{\int_{t_0}^{t} (g(s) + r(s))ds}
\]
holds.
(3)

**Proof.** Denote the right parts of (1), (2) by \( \nu(t) \).

Function \( \nu(t) \) is increasing for \( t \geq t_0 \) and \( \max_{s \in [t-h, t]} \nu(s) = \nu(t) \).

Then from inequality (1) we get

\[
\nu(t) = C + \int_{t_0}^{t} g(s)x(s)ds + \int_{t_0}^{t} r(s) \max_{\zeta \in [s-h,s]} x(\zeta)ds \leq
\]

\[
\max_{s \in [t_0 - h, t_0]} \varphi(s) + \int_{t_0}^{t} g(s)\nu(s)ds + \int_{t_0}^{t} r(s) \max_{\zeta \in [s-h,s]} \nu(\zeta)ds.
\]

Therefore

\[
\nu(t) \leq \max_{s \in [t_0 - h, t_0]} \varphi(s) + \int_{t_0}^{t} g(s) \max_{\zeta \in [s-h,s]} \nu(\zeta)ds +
\]

\[
+ \int_{t_0}^{t} r(s) \max_{\zeta \in [s-h,s]} \nu(\zeta)ds.
\]

Then

\[
\nu(t) \leq \max_{s \in [t_0 - h, t_0]} \varphi(s) + \int_{t_0}^{t} (g(s) + r(s))\nu(s)ds.
\]
(4)

From inequality (4) and Gronwall-Bellman inequality we obtain

\[
\nu(t) \leq \max_{s \in [t_0 - h, t_0]} \varphi(s) e^{\int_{t_0}^{t} (g(s) + r(s))ds}
\]
(5)

Inequality \( x(t) \leq \nu(t) \) and inequality (5) imply the validity of (3). \( \square \)
Corollary. Let the conditions of Theorem 1 be satisfied for \( h = 0 \) and \( \varphi(t) \equiv C \). Then the inequality (3) reduces to the classical Gronwall-Bellman inequality.

**Theorem 2:** Let following conditions be satisfied:
1) Function \( a \in C(\lbrack t_0, \infty), (0, \infty) \) is an increasing function.
2) Functions \( g, r \in C(\lbrack t_0, \infty), R_+ \) and function \( \varphi \in C(\lbrack t_0 - h, t_0], R_+ \).
3) Function \( x \in C(\lbrack t_0, \infty), R_+ \) and it satisfies the inequalities

\[
x(t) \leq a(t) + \int_{t_0}^{t} g(s)x(s)\,ds + \int_{t_0}^{t} r(s)\max_{\zeta \in [s-h,s]} x(\zeta)\,ds \quad \text{for } t \geq t_0 ,
\]
\[
x(s) \leq \varphi(s) \quad \text{for } s \in [t_0 - h, t_0] , \text{ where } h = \text{const} \geq 0 , \quad \varphi(t_0) = a(t_0).
\]

Then the inequality

\[
x(t) \leq a(t) \max_{s \in [t_0 - h, t_0]} \varphi(s) \int_{t_0}^{t} (g(s) + r(s))\,ds
\]

for \( t \geq t_0 \) holds.

**Proof.** From inequalities (6), (7) we obtain

\[
\frac{x(t)}{a(t)} \leq 1 + \int_{t_0}^{t} \frac{g(s)}{a(t)}x(s)\,ds + \int_{t_0}^{t} \frac{r(s)}{a(t)} \max_{\zeta \in [s-h,s]} x(\zeta)\,ds
\]

for \( t \geq t_0 ,\)

\[
\frac{x(t)}{a(t + h)} \leq \frac{\varphi(t)}{a(t + h)} \quad \text{for } t \in [t_0 - h, t_0].
\]

We will prove that

\[
\frac{x(\zeta)}{a(\zeta)} \leq \frac{x(\zeta)}{a(\zeta)} \quad \text{for } \zeta \in [s-h,s] ,
\]

\[
\frac{x(\zeta)}{a(\zeta)} \leq \frac{x(\zeta)}{a(s)} \quad \text{for } \zeta \in [s-h,s] ,
\]

since \( a(s) \geq a(\zeta) \).

Therefore
Define $p(t) = \begin{cases} \frac{x(t)}{a(t)}, & t \geq t_0 \\ \frac{x(t)}{a(t + h)}, & t \in [t_0 - h, t_0] \end{cases}$.

From definition (11) and inequalities (9), (10) according to Theorem 1 we obtain

$$p(t) \leq 1. e^{t_0},$$

$$x(t) \leq a(t) \frac{\max_{s \in [t_0 - h, t_0]} \varphi(s)}{a(t_0)} e^{t_0}.$$  \hfill \Box

3. Applications

Consider the system of differential equations equations with "maximum"

$$x' = f(t, x(t), \max_{s \in [t - h, t]} x(s)) \quad \text{for} \quad t \geq t_0$$

with initial condition

$$x(t) = \varphi(t) \quad \text{for} \quad t \in [t_0 - h, t_0],$$

where

$$x \in \mathbb{R}^n, \quad h = \text{const} \geq 0, \quad \varphi : [t_0 - h, t_0] \to \mathbb{R}^n, \quad f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n.$$

We will apply the above solved integral inequalities to the initial value problem (12), (13) for some qualitative investigations.

**Lemma 1.** Let $u, v \in C([t_0 - h, \infty), \mathbb{R}^n)$. Then for all $s \geq t_0$ the inequality

$$\left\| \max_{\zeta \in [s-h,s]} u(\zeta) - \max_{\zeta \in [s-h,s]} v(\zeta) \right\| \leq \max_{\zeta \in [s-h,s]} \| u(\zeta) - v(\zeta) \|$$

holds.

**Proof.** Let $n = 1$ and $t \geq t_0$.

There exists a point $P \in [t - h, t]$ such that $\max_{s \in [t-h,t]} u(s) = u(P)$. 

$$s \in [t-h,t]$$
Linear integral inequalities involving “maxima”

Then
\[
\max_{\zeta \in [t-h, t]} u(\zeta) - \max_{\zeta \in [t-h, t]} v(\zeta) = \\
\| u(P) - \max_{\zeta \in [t-h, t]} v(\zeta) \| \leq \| u(p) - v(p) \| \leq \max_{\zeta \in [t-h, t]} \| u(\zeta) - v(\zeta) \|.
\]

Now let consider an arbitrary natural number \( n > 1 \).

Then from above proved inequality we have
\[
\left\| \max_{\zeta \in [t-h, t]} u(\zeta) - \max_{\zeta \in [t-h, t]} v(\zeta) \right\| = \sum_{t=1}^{n} \max_{\zeta \in [t-h, t]} \left\| u_{i}(\zeta) - v_{i}(\zeta) \right\| \\
\leq \sum_{t=1}^{n} \max_{\zeta \in [t-h, t]} \left\| u_{i}(\zeta) - v_{i}(\zeta) \right\|
\]
\[
\leq \max_{\zeta \in [t-h, t]} \sum_{i=1}^{n} \left\| u_{i}(\zeta) - v_{i}(\zeta) \right\| = \max_{\zeta \in [t-h, t]} \left\| u(\zeta) - v(\zeta) \right\|
\]

\[\square\]

A) Uniqueness of the solution

**Theorem 3.** Let following conditions be satisfied:

1) Function \( f \in C([t_0, \infty) \times R^n \times R^n, R_+) \) and satisfies for \( t \geq t_0 \) and 
\( x_i, y_i \in R^n \), \( i = 1, 2 \), the condition
\[
\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq g(t) \| x_1 - x_2 \| + r(t) \| y_1 - y_2 \|
\]
\[
\| f(t, x_1, y_1) - f(t, x_2, y_2) \| \leq g(t) \| x_1 - x_2 \| + r(t) \| y_1 - y_2 \|,
\]
where \( g(t), r(t) \in C([t_0, \infty), R_+) \).

2) For any initial function \( \varphi \in C([t_0 - h, t_0], R^n) \) the initial value problem (12), (13) has at least one solution defined for \( t \geq t_0 \).

Then the initial value problem (12), (13) has a unique solution.

**Proof.** Let \( \varphi \in C([t_0 - h, t_0], R^n) \) be a fixed initial function. Assume that there exist two different solutions \( u(t), v(t) \) of the initial value problem (12), (13). Both functions \( u(t) \) and \( v(t) \) satisfy the integral equalities
\[ u(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, u(s), \max_{\zeta \in [s-h,s]} u(\zeta)) \, ds, \]
\[ v(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, v(s), \max_{\zeta \in [s-h,s]} v(\zeta)) \, ds \quad \text{for} \quad t \geq t_0 \]
and \( v(t) = u(t) = \varphi(t) \) for \( t \in [t_0 - h, t_0] \).

Then the difference of both solutions \( u(t) \) and \( v(t) \) satisfies the inequalities
\[
\|u(t) - v(t)\| \leq 0 + \int_{t_0}^{t} g(s) \|u(s) - v(s)\| ds + t \int_{t_0}^{t} r(s) \left( \max_{\zeta \in [s-h,s]} u(\zeta) - \max_{\zeta \in [s-h,s]} v(\zeta) \right) ds, \quad t \geq t_0 \tag{14}
\]
and \( \|u(t) - v(t)\| = 0 \) for \( t \in [t_0 - h, t_0] \). \tag{15}

If we set \( \|u(t) - v(t)\| = p(t) \) for \( t \in [t_0 - h, \infty) \), then according to Lemma 1, inequality (14) and equality (15) we obtain
\[
p(t) \leq 0 + \int_{t_0}^{t} g(s)p(s)ds + \int_{t_0}^{t} r(s) \max_{\zeta \in [s-h,s]} p(\zeta)ds \quad \text{for} \quad t \geq t_0 \quad \text{and} \quad p(t) = 0
\]
for \( t \in [t_0 - h, t_0] \). According to Theorem 1, we have
\[
p(t) = 0 \int_{t_0}^{t} (g(s) + r(s)) ds = 0
\]
Therefore \( \|u(t) - v(t)\| = 0 \) for \( t \geq t_0 \) or \( u(t) \equiv v(t) \).

\[ \Box \]

B) Continuous dependence of the solution on the initial conditions

**Theorem 4.** Let following conditions be satisfied:

1) \( f \in C([t_0, \infty) \times R^n \times R^n, R_+) \) and satisfies the Lipshitz condition for \( t \geq t_0, x_i, y_i \in R^n, i = 1, 2 \):

Linear integral inequalities involving “maxima”

\[ \|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq g(t) \|x_1 - x_2\| + r(t) \|y_1 - y_2\|, \text{ where } g(t), r(t) \in C([t_0, \infty), R_+). \]

2) The initial value problem (12), (13) has a solution for every initial function \( \varphi \in C([t_0 - h, t_0], R^n) \).

Then for every number \( \varepsilon > 0 \) and every \( T < \infty \) there exists \( \delta = \delta(\varepsilon, t_0, T) > 0 \) such that the inequality

\[ \|\varphi(s) - \psi(s)\| \leq \delta \text{ for } s \in [t_0 - h, t_0] \text{ implies } \|x(t) - y(t)\| < \varepsilon \text{ on } (t_0, T), \]

where \( x(t), y(t) \) are solutions of (12), (13) with initial functions

\( \varphi(t), \psi(t) \in C([t_0 - h, t_0], R^n) \) correspondingly.

**Proof.** Let \( \varepsilon > 0 \) be an arbitrary number and \( \varphi(t), \psi(t) \) be two different initial functions. Choose a number \( T \)

\( \delta < \varepsilon \exp(-\int_{t_0}^T (g(s) + r(s)) ds) \).

We consider the difference between solutions \( x(t), y(t) \) and define the function \( p(t) : [t_0 - h, T] \to R_+ \) by the equality \( p(t) = \|x(t) - y(t)\| \).

Hence,

\[ \|x(t) - y(t)\| \leq \|x(t_0) - y(t_0)\| + \int_{t_0}^T g(s) \|x(s) - y(s)\| ds + \]

\[ + \int_{t_0}^T r(s) \max_{\zeta \in [s-h,s]} x(\zeta) - \max_{\zeta \in [s-h,s]} y(\zeta) ds, t \geq t_0 \]

\[ \|x(t) - y(t)\| = \|\varphi(s) - \psi(s)\|, t \in [t_0 - h, t_0]. \]

According to Theorem 1 we obtain

\[ p(t) \leq \]

\[ \int_{t_0}^T (g(s) + r(s)) ds \]

\[ \leq \max_{t \in [t_0 - h, t_0]} \|\varphi(t) - \psi(t)\| e^{\int_{t_0}^T g(s) + r(s) ds} \leq \delta e^{\int_{t_0}^T g(s) + r(s) ds} < \varepsilon. \]

\( \Box \)
Acknowledgments

The authors would like to thank Prof. S. G. Hristova for the statement of the problem and suggestions on the proofs in the paper.

References


Faculty of Mathematics and Informatics
Plovdiv University “Paisii Hilendariski”
24 Tsar Asen, 4000 Plovdiv
BULGARIA
E-mail: stela_87s@abv.bg,
E-mail: kremena_87@yahoo.com
Boundary value problem for a class of fourth-order partial differential equations of mixed type

G. P. Paskalev

Abstract

In this paper we prove existence and uniqueness of the generalized solution of a local boundary value problem for a class of fourth-order partial differential equations of mixed type in cylindrical domain.

1. Introduction

Let $D \subset \mathbb{R}^n$, $n \geq 1$ be a bounded domain with a boundary $\partial D$. Denote:

$x = (x_1, x_2, x_3, \ldots, x_n), G = D \times (0, T), \Gamma = D \times (0, T), T > 0.$

Suppose that $\Gamma$ is smooth and let us consider in $G$ the equation

$L u \equiv P_4(t, x) u - M_4(x) u + [c(t, x) - C] u = f(t, x), \quad (1)$

where

$P_4(t, x) u \equiv \sum_{i=1}^{4} k_i(t, x) D_i^4 u; \quad M_4(x) u \equiv \sum_{|\alpha|=|\beta|=2} D_x^\alpha [a_\alpha \beta(x) D_\beta^\beta u];$

$D_i^4 u(t, x) = \frac{\partial^4}{\partial x_i^4} u(t, x); \quad D_\alpha^\alpha u(t, x) = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} u(t, x);$

$\alpha_i \geq 0$ are integer, $C=\text{const}>0$ and the coefficients $k_i(t, x), c(t, x), a^{\alpha \beta}(x)$

$(a^{\alpha \beta}(x) \equiv a^{\beta \alpha}(x) \, \forall \alpha, \beta)$ are infinitely smooth functions in $G$.

Suppose that the condition

$k_4(T, x) = k_4(0, x) = 0 \quad \forall x \in \bar{D}$

is satisfied and

2000 Mathematics Subject Classification: 35G15, 35G10.

Key words and phrases: fourth-order equation, local boundary conditions, anisotropic Sobolev spaces, a priori estimates

Received June 15, 2009.
\[ \sum_{l\alpha=|\beta|=2} \xi^\alpha a_{\alpha\beta}(x) \xi^\beta \geq C_0 |\xi|^2 m \quad \forall \xi \in R^n \quad \forall x \in \bar{D}, \]

where \( C_0 = const > 0 \) and \( \alpha, \beta \) are multi indexes.

The equation (1) is a mixed type equation in \( G \cup \Gamma \) and on the bottoms of the cylindrical domain the equation is parabolic.

### 2. Boundary conditions and function spaces

Consider the following boundary value problem. To find a solution of equation (1) in \( G \), satisfying the boundary conditions:

\[ D^\alpha_x u \big|_{\Gamma} = 0; \quad \forall \alpha \leq 1 \]
\[ u(0, x) = 0, \quad D_t u(0, x) = 0, \quad D^2_t u(T, x) = 0; \]

Let \( \tilde{C}^\infty(\bar{G}) \) be the space of infinitely smooth in \( G \) functions, satisfying the boundary conditions (2) and (3) and let \( \tilde{C}^\infty_*(\bar{G}) \) be the corresponding space of infinitely smooth in \( G \) functions, satisfying the adjoint to (2) and (3) boundary conditions:

\[ D^\alpha_t v \big|_{\Gamma} = 0; \quad \forall \alpha \leq 1 \]
\[ D^2_t v(0, x) = 0, \quad v(T, x) = 0, \quad D_t v(T, x) = 0; \]

If \( p \geq 1, q \geq 1 \) are integer numbers, let us define the space \( H^{p,q}_{t,x}(G) \) as the closure of \( \tilde{C}^\infty_*(\bar{G}) \) with respect to the norm

\[ \|u\|^2_{p,q} = \int_G \sum_{q_i + p|\alpha| \leq p} (D^i_t D^\alpha_x u)^2 \, dt \, dx \]

and the space \( H^{p,q}_{t,x,*}(G) \) as the closure of \( \tilde{C}^\infty_*(\bar{G}) \) with respect to the same norm.

The scalar product of the space \( L^2(G) \equiv H^{0,0}_{t,x}(G) \) we shall denote by \( (.,.)_0 \).

**Definition.** A function \( u \in H^{3,2}_{t,x}(G) \) is called a generalized solution for the problem (1)-(3) if

\[ (u, L^* v)_0 = (f, v)_0 \quad \forall v \in \tilde{C}^\infty_*(\bar{G}) \]

### 3. Results

**Theorem 1.** Let the following condition is satisfied:

\[ 2k_3(t, x) - D_t k_4(t, x) \geq \delta = const > 0 \quad \forall (t, x) \in \bar{G}. \]

Then for any function \( f \in L^2(G) \) there exists a generalized solution for the problem (1)-(3).

**Theorem 2.** Let the following condition is satisfied:

\[ 2k_3(t, x) - 7k_4(t, x) \geq \delta_1 = const > 0 \quad \forall (t, x) \in \bar{G}. \]

Then the problem (1)-(3) can have no more than one generalized solution.
4. Proofs

**Proof of theorem 1.** Let we define the function $\Phi(t) = \frac{(t - T)^3}{3!} + \gamma_0$, where $\gamma_0$ is a sufficiently large constant, which we shall refine below. For $u \in \tilde{C}^\infty(\tilde{G})$ we define the operator

$$R(t) = \sum_{i=1}^{3} (\frac{3}{i})D_t^{3-i}\Phi(t)D_t^iu + \Phi(t)D_t^3u.$$  \hspace{1cm} (5)

If the constant $C$ is sufficiently large, by integration by parts, using the Garding’s inequality and appending theorem 10.2 from [6], we obtain the following estimate:

$$(Lu, R(t)u)_0 \geq const.\ |u|_{B^2}^2, \forall u \in \tilde{C}^\infty(\tilde{G}).$$  \hspace{1cm} (6)

For any function $v \in \tilde{C}^\infty(\tilde{G})$ consider the problem

$$R(t)u = v$$  \hspace{1cm} (7)

$$D^\alpha Tu \bigg|_{\Gamma} = 0; 1\alpha 1 \leq$$  \hspace{1cm} (8)

$$u(0, x) = 0, D_tu(0, x) = 0, D^2_tu(T, x) = 0;$$  \hspace{1cm} (9)

and using the Cauchy method to build a partial solution $g(t, x)$ to (7) [5, p.459], we reduce the question for solvability of the problem (7)-(9) to the solvability of the system, obtained from the boundary conditions (9):

$$\begin{pmatrix} u_1(0) & u_2(0) & u_3(0) \\ u'_1(0) & u'_2(0) & u'_3(0) \\ u''_1(T) & u''_2(T) & u''_3(T) \end{pmatrix} \begin{pmatrix} C_1(x) \\ C_2(x) \\ C_3(x) \end{pmatrix} = \begin{pmatrix} -g(0, x) \\ -D_tg(0, x) \\ -D^2_tv(0, x) \end{pmatrix}.$$  \hspace{1cm} (10)

where $u_1(t), u_2(t), u_3(t)$ is the corresponding fundamental system of solutions. Clearly, we can take $u_3(t) \equiv 1$ and now we can prove our statement only for $u_1(t), u_2(t).$ Consider the equation

$$2\left[\frac{(t - T)^3}{3!} + \gamma_0\right]u''' + 3\left(\frac{(t - T)^2}{2!}\right)u'' + 3(t - T)u' = 0;$$

Denoting $y(t, \gamma_0) = u'(t)$, we obtain

$$2\left[\frac{(t - T)^3}{3!} + \gamma_0\right]y''(t, \gamma_0) + 3\left(\frac{(t - T)^2}{2!}\right)y'(t, \gamma_0) + 3(t - T)y(t, \gamma_0) = 0;$$  \hspace{1cm} (10)

We shall prove that if the constant $\gamma_0$ is sufficiently large, then for any fundamental system of solutions $y_1(t, \gamma_0), y_2(t, \gamma_0)$ to this equation it follows that
If we integrate (10) from 0 to T for \( y = y_j(t, \gamma_0), \ j = 1, 2 \), then obtain

\[
2 \int_0^T \left[ \frac{(t-T)^3}{3!} + \gamma_0 \right] y''(t, \gamma_0) dt + 3 \int_0^T \frac{(t-T)^2}{2!} y'(t, \gamma_0) dt +
\]

\[
+ 3 \int_0^T (t-T)y(t, \gamma_0) dt = 0; \ j = 1, 2,
\]

Now by integration by parts it obtains that

\[
2 \gamma_0 y_j'(T, \gamma_0) - 2[ \gamma_0 - \frac{T^3}{3!}] y_j(0, \gamma_0) - \frac{T^2}{2!} y_j(0, \gamma_0) +
\]

\[
+ 2 \int_0^T (t-T)y_j(t, \gamma_0) dt = 0, \ j = 1, 2,
\]

from where

\[
y_j'(T, \gamma_0) = \left[ 1 - \frac{1}{\gamma_0} \frac{T^3}{3!} \right] y_j(0, \gamma_0) + \frac{1}{2 \gamma_0} \frac{T^2}{2!} y_j(0, \gamma_0) -
\]

\[
- \frac{1}{\gamma_0} \int_0^T (t-T)y_j(t, \gamma_0) dt, \ j = 1, 2,
\]

From here it follows that

\[
\det \begin{pmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ y_1'(\gamma_0) & y_2'(\gamma_0) \end{pmatrix} = \left[ 1 - \frac{1}{\gamma_0} \frac{T^3}{3!} \right] \det \begin{pmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ y_1'(0, \gamma_0) & y_2'(0, \gamma_0) \end{pmatrix}
\]

\[
+ \frac{1}{2 \gamma_0} \frac{T^2}{2!} \det \begin{pmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ y_1(0, \gamma_0) & y_2(0, \gamma_0) \end{pmatrix}
\]

\[
+ \frac{T}{\gamma_0} \det \begin{pmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ (\theta_1 - T)y_1(\theta_1, \gamma_0) & (\theta_2 - T)y_2(\theta_2, \gamma_0) \end{pmatrix}
\]

\[
= \left[ 1 - \frac{1}{\gamma_0} \frac{T^3}{3!} \right] \Delta_1 + \frac{1}{2 \gamma_0} \frac{T^2}{2!} \Delta_2 + \frac{T}{\gamma_0} \Delta_3,
\]

where \( \theta_1, \theta_2 \in (0, T) \) are the corresponding values from the mean value integral theorem.

But \( \Delta_1 \) is the Wronskian, constructed by the fundamental system of solutions \( y_1(t, x), y_2(t, x) \) and hence \( \Delta_1 \neq 0 \). From the other hand \( \Delta_2 = 0 \) and the solutions
Boundary value problem for a class of fourth-order partial...

\( y_1(t,x), y_2(t,x) \) are bounded with respect to the parameter \( \gamma_0 \) because when the parameter \( \gamma_0 \) tends to infinity, the coefficients of the equation (10) tend to the coefficients of the equation \( y'' = 0 \). Hence \( \Delta_1 \) also is bounded with respect to \( \gamma_0 \). Then from (12) it follows that

\[
\text{sign} \left[ \begin{vmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ y_1'(0, \gamma_0) & y_2'(0, \gamma_0) \end{vmatrix} \right] = \text{sign} \left[ \begin{vmatrix} y_1(0, \gamma_0) & y_2(0, \gamma_0) \\ y_1'(0, \gamma_0) & y_2'(0, \gamma_0) \end{vmatrix} \right] \neq 0
\]

for sufficiently large parameter \( \gamma_0 \). The inequality (11) is proved.

Let we denote by \( H_{t,x}^{(3,2)}(G) \) the space with negative norm of Lax, adjoint to \( H_{t,x}^{(3,2)}(G) \). From the estimate (6) and if \( u(t,x) \) is the solution of the problem (7)-(9) for any fixed function \( v \in C_{*}^{\infty}(G) \) we have

\[
\| L^* v \|_{(3,2)} \geq \| L^* v, u \|_0 = (v, Lu)_0 = (R u, Lu)_0 \geq \text{const.} \| u \|_{3,2}^2,
\]

from where

\[
\| L^* v \|_{(3,2)} \geq \text{const.} \| v \|_0 \quad \forall v \in \tilde{C}_{*}^{\infty}(G),
\]

because from the equality \( R(t) u = v \) it follows that \( \| u \|_{s-1,m} \leq \| v \|_0 \). Hence there exists a function \( u \in H_{t,x}^{3,2}(G) \) for which (4) is true ([1]). The theorem is proved.

**Proof of theorem 2:** Using the function \( R(t) = \frac{t^3}{3!} + \gamma_0 \) in the definition (5) we obtain the following a priori estimate

\[
\| L u \|_{(3,2)} \geq \text{const.} \| u \|_0 \quad \forall u \in \tilde{C}_{*}^{\infty}(G),
\]

from where it follows uniqueness of the generalized solution \( u \in H_{t,x}^{3,2}(G) \) ([2]). The theorem is proved.

**5. Example**

Let \( D = \{(x_1, x_2) / x_1^2 + x_2^2 < R^2 \}, G = D \times (0,1) \) where \( R=\text{const}\geq0 \) and \( \Omega \in C^{\infty}[0,1], \Omega(0) = \Omega(1) = 0, f \in L^2(G) \). Consider the problem

\[
\Omega(t) D_t^4 u + A D_t^3 u - D_{x_1}^4 u - D_{x_2}^4 u - C u = f(t,x) \text{ in } G, \tag{13}
\]

\[
D_2^0 u \big|_{\Gamma} = 0; \text{ } |\alpha| \leq 1 \tag{14}
\]

\[
u(0,x) = 0, D_t u(0,x) = 0, D_t^2 u(1,x) = 0; \tag{15}
\]

where \( A=\text{const}\geq0, C=\text{const}\geq0 \). Then we have \( k_4(t,x) \equiv \Omega(t), k_3(t,x) \equiv A \).
\[ k_{1,2}(t,x) \equiv 0, c(t,x) \equiv 0, a^\alpha(x) \equiv 1 \]

for \( \alpha = \beta \equiv (2,0), (0,2) \) else \( a^\alpha(x) \equiv 0 \).

If the constants A,C are sufficiently large, then the conditions of the theorems 1,2 are satisfied. Hence the problem (13)-(15) has a unique generalized solution \( u \in H_{t,x}^{3,2}(G) \).

For problems with nonlocal boundary value conditions a priori estimates similar to (5), (6) are used in [3,7,8]. A local problem for higher order partial differential equation is considered in [4]. The smoothness of the solutions of the boundary value problem (1)-(3) we shall consider in another paper.

References


Department of Mathematics
Technical University, Plovdiv
25, Tsanko Dyustabanov Str.
4000 Plovdiv
BULGARIA

e-mail: g.p.paskalev@abv.bg
Smoothness of the solutions of boundary value problem for a class of fourth-order partial differential equations of mixed type

G. P. Paskalev

Abstract

In this paper we give sufficient conditions for existence of smooth and classical solution of a local boundary value problem for a class of fourth-order partial differential equations of mixed type in cylindrical domain.

1. Introduction

The present paper continues the considerations from [3], where existence and uniqueness of the generalized solution are obtained.

Let $D \subseteq \mathbb{R}^n, n \geq 1$ be a bounded domain with a boundary $\partial D$. Denote:

$$x = (x_1, x_2, x_3, \ldots, x_n), G = D \times (0, T), \Gamma = D \times (0, T), T > 0.$$

Suppose that $\Gamma$ is smooth and let us consider in $G$ the equation

$$L u \equiv P_4(t, x)u - M_4(x)u + [c(t, x) - C]u = f(t, x), \quad (1)$$

where

$$P_4(t, x)u \equiv \sum_{i=1}^{4} k_i(t, x)D_i^1 u; \quad M_4(x)u \equiv \sum_{|\alpha| = |\beta| = 2} D_x^\alpha[a_\alpha\beta(x)D_x^\beta u];$$

$C = Const > 0$ and the coefficients

$$k_i(t, x), c(t, x), a_\alpha^\beta(x) (a_\alpha^\beta(x) \equiv a_\beta^\alpha(x) \forall \alpha, \beta)$$

are infinitely smooth functions in $\bar{G}$.

Suppose that the conditions

2000 Mathematics Subject Classification: 35G15, 35G10.

Key words and phrases: fourth-order equation, local boundary conditions, anisotropic Sobolev space, smooth and classical solution.

Received June 15, 2009.
$k_4(T, x) = k_4(0, x) = 0 \ \forall x \in \bar{D}$, $k_1(t, x) \equiv k_2(t, x) \equiv 0$, $\forall (t, x) \in \bar{G}$;

are satisfied and

$$
\sum_{i\alpha = |\beta| = 2} \xi^{\alpha} a_{\alpha\beta}(x) \xi^{\beta} \geq C_0 |\xi|^m \ \forall \xi \in \mathbb{R}^n \ \forall x \in \bar{D},
$$

where $C_0 = const > 0$ and $\alpha, \beta$ are multi indexes.

2. Boundary conditions and function spaces

Consider the following boundary value problem: To find a solution of equation (1) in $G$, satisfying the boundary conditions:

$$
D_x^{\alpha} u \big|_{\Gamma} = 0; |\alpha| \leq 1
$$

$$
u(0, x) = 0, D_t u(0, x) = 0, D_t^2 u(T, x) = 0;
$$

The spaces $\bar{C}^\infty(\bar{G})$ and $\bar{C}^{\infty}_*(\bar{G})$ are defined in [3]. If $p \geq 1$ and $q \geq 1$ are integer numbers, define the space $W^{p, q}(G)$ as a set of functions $u \in L_2(G)$ which have generalized derivatives

$$
D^{\alpha}_x D^{\beta}_t u \in L_2(G) \forall (i, \alpha) : \frac{i}{p} + \frac{|\alpha|}{q} \leq 1.
$$

By definition $W^{p, q}(G)$ is a space with a norm

$$
\|u\|_{p, q}^2 = \int_G \sum_{q_i + p|\alpha| \leq pq} (D^{\alpha}_x D^{\beta}_t u)^2 \, dt \, dx
$$

If $p \geq 1, q \geq 1$ are integer numbers, define the space $H^{p, q}(G)$ as the closure of the function space $\bar{C}^\infty(\bar{G})$ with respect to the norm (4) and the space $H^{p, q}_*(G)$ as the closure of $\bar{C}^{\infty}_*(\bar{G})$ with respect to the same norm.

From definitions it follows that $H^{p, q}_*(G) \subset W^{p, q}(G)$.

3. Main result

**Theorem.** Let $l \geq 1$ is an integer number and

(a) $f \in W^{3, 2l}_{l, x}(G)$,

(b) $D^{-1}_t f(0, x) = 0, D^1_t f(0, x) = 0, D^{i+1}_t f(T, x) = 0$;

$i = 1 + 3 j, j = 0, l - 1$;

almost everywhere in $D$.

(c) $2k_3(t, x) + r D_t k_4(t, x) \geq \delta = const > 0$,

$r = 6p - 1, p = \overline{0, l}, \quad r = 6p - 7, p = \overline{0, l},$
Smoothness of the solutions of boundary value problem for...

Then the generalized solution $u(t,x)$ of the problem (1)-(3) belongs to the space $W^{3(l+1),2(l+1)}_{t,x}(G)$ and

\[ D_t^{i-1}u(0,x) = 0, \quad D_t^i u(0,x) = 0, \quad D_t^{i+1}u(T,x) = 0; \quad i = 1 + 3j, \quad j = 0,1; \]

almost everywhere in $D$.

4. Proof

In order to prove this theorem we apply the schema, used in [1]. In the case $l=1$ we prove that if $u$ is the generalized solution for the problem (1)-(3) then $D_t^3u$ is generalized solution of the same problem for the equation $L_w = f_1$, where

\[
L_w \equiv k_4(t,x)D_t^4w + [k_3(t,x) + 3D_t k_4(t,x)]D_t^3w + \\
+ [3D_t^2k_4(t,x) + 3D_t k_3(t,x)]D_t^2w + \\
+ [3D_t^2k_3(t,x) - N]w - \sum_{\alpha \leq |\beta| = 2} D_x^\alpha [a_{\alpha \beta}(x)D_x^\beta w]
\]

and $f_1(t,x) = D_t^3\{f(t,x) - [c(t,x) - C]u - Nu\}$, where $N$ is a sufficiently large positive constant.

Consider the problem

\[
L_1w = f_1 \text{ in } G
\]

\[
D_x^\alpha w \big|_{\Gamma} = 0; \quad \alpha \leq 1
\]

\[
w(0,x) = 0, \quad D_t w(0,x) = 0, \quad D_t^2 w(T,x) = 0;
\]

The conditions of theorems 1,2 from [3] are true and hence the problem (5)-(7) has a unique generalized solution $w \in W^{3,2}_{t,x}(G)$.

If now $\zeta \in \tilde{C}_w^\infty(G)$ is an arbitrary element, consider the function

\[
v(t,x) = \int_0^T \left( \frac{(t-\tau)^2}{2} \zeta(\tau,x) d\tau + \frac{(t-T)^2}{2} \int_0^T \zeta(\tau,x) d\tau \right)
\]

Then

(a) $D_t^3v(t,x) = \zeta(t,x) \quad \forall (t,x) \in G$, (b) $v \in \tilde{C}_w^\infty(G)$.

Hence we have

\[
(w, L_1^*v)_0 = (f_1, v)_0
\]

where
\[ L_1^*v = D_t^3\{ k_4(t, x)v \} - D_t^3\{ [k_3(t, x) + 3D_tk_4(t, x)]v \} + \]
\[ + D_t^2\{ [3k_3(t, x) + 3D_t^2k_4(t, x)]v \} - \]
\[ -D_t\{ [D_t^3k_4(t, x) + 3D_t^2k_3(t, x)] \} - \]
\[ \sum_{|\alpha|=|\beta|=2} D_x^\alpha[a_{\alpha\beta}(x)D_x^\beta v] + [3D_t^3k_3(t, x) - N]v. \]

By integration by parts we obtain
\[ (f_1, v)_0 = (D_t^3\{ f - [c(t, x) - C]u - Nu \}, v)_0 = \]
\[ = -(f - [c(t, x) - C]u - Nu, D_t^3v)_0. \] (9)

Let we pose
\[ \Phi(t, x) = \frac{1}{2} \int_0^T \frac{(t - \tau)^2}{2} w(\tau, x) d\tau + \frac{t^2}{2} \int_0^T w(\tau, x) d\tau. \]

Then \( \Phi \in W_{t, x}^{3, 2}(G) \) and \( D_t^3\Phi(t, x) = w(t, x) \) in \( G \) and now we have
\[ (D_t^3\Phi, L_1^*v)_0 = -(f - [c(t, x) - C]u - Nu, v)_0 \] (10)

Let we integrate by parts to the left in the last equality.
\[ (D_t^3\Phi, L_1^*v)_0 = -(\Phi, D_t^3\{ D_t^4[k_4(t, x)v] - D_t^3[(k_3(t, x) + 3D_tk_4(t, x))]v \} + \]
\[ + D_t^2[(3k_3(t, x) + 3D_tk_4(t, x))]v] - D_t[(D_t^3k_4(t, x) + 3D_t^2k_3(t, x))]v - \]
\[ - \sum_{|\alpha|=|\beta|=2} D_x^\alpha[a_{\alpha\beta}(x)D_x^\beta v] + [D_t^3k_3(t, x) - N]v)_0 = \]
\[ = -(\Phi, D_t^3\{ D_t[k_4(t, x).D_t^3v] - k_3(t, x)D_t^3v \} - \]
\[ - \sum_{|\alpha|=|\beta|=2} D_x^\alpha[a_{\alpha\beta}(x)D_x^\beta D_t^3v] - ND_t^3v)_0 = \]
\[ = -(f - [c(t, x) - C]u - Nu, \zeta)_0 \] (11)

where the last equality follows from (10).

Consider the operator
\[ L_2\psi = k_4(t, x)D_t^4\psi + k_3(t, x)D_t^3\psi - \sum_{|\alpha|=|\beta|=2} D_x^\alpha[a_{\alpha\beta}(x)D_x^\beta \psi] - N\psi. \]

and his adjoint
\[ L_2^\ast \zeta = D_t^4 [k_4(t, x) \zeta] - D_t^3 [k_3(t, x) \zeta] - \sum_{\alpha \beta} D_x^\alpha a_{\alpha \beta}(x) D_x^\beta \zeta - N \zeta. \]

Now from (12) it follows that \((w, L_1^* v)_0 = -\langle \Phi, L_2^* \zeta \rangle_0\) and from (8),(9) we obtain
\[ (\Phi, L_2^* \zeta)_0 = (f - \lfloor c(t, x) - C \rfloor u - Nu, \zeta)_0 \quad (12) \]

From the other hand \((u, L_2^* \zeta)_0 = (f, \zeta)_0 \quad \forall v \in \tilde{G}_\infty^{\ast}(\tilde{G})\), from where
\[ (u, L_2^* \zeta)_0 = (f - \lfloor c(t, x) - C \rfloor u - Nu, \zeta)_0 \quad (13) \]

Using the uniqueness theorem, we conclude from (12) and (13) that if the constant \(N\) is sufficiently large, then \(\Phi = u\) almost everywhere in \(G\). Hence \(w = D_t^3 u\) also almost everywhere in \(G\) and then \(D_t^3 u \in W_{t,x}^{3,2}(G)\).

But the equation \(L u = f\) is satisfied in weak sense for each function \(\zeta \in C_0^\infty(G)\) the equality \(A(u, \zeta) = (f_2, \zeta)_0\) is fulfilled, where
\[
A(u, \zeta) = \int_G \sum_{\alpha \beta} a_{\alpha \beta}(x) D_x^\alpha u D_x^\beta \zeta dt dx, f_2 = \]
\[
= f - \lfloor c(t, x) - C \rfloor u - k_3 D_t^3 u - k_4 D_t^4 u. \]

From [5], Theorem 3 it follows that \(u \in W_{t,x}^{0,4}(G)\). Now from the estimates for the mixed derivatives (point 10.2 from [6]) we conclude that \(u \in W_{t,x}^{0,4}(G)\). From the equalities \(w(0, x) = 0, D_t w(0, x) = 0, D_t^2 w(T, x) = 0\); almost everywhere in \(D\), we obtain that \(D_t^3 u(0, x) = 0, D_t^4 u(0, x) = 0, D_t^5 u(T, x) = 0\).

The theorem is proved in the case when \(l=1\).

Let now \(l_0 \geq 1\) is a fixed integer number and suppose that the theorem is true for \(l = l_0\) and that the conditions are true for \(l = l_0 + 1\) Hence the problem (1)-(3) has a unique solution \(u \in W_{t,x}^{3(l_0 + 1),2(l_0 + 1)}(G)\) such that
\[ D_t^{i-1} u(0, x) = 0, D_t^i u(0, x) = 0, D_t^{i+1} u(T, x) = 0; i = 1 + 3 j, j = 1, l_0; \]
almost everywhere in \(D\).

But if \(f, u \in W_{t,x}^{3(l_0 + 1),2(l_0 + 1)}(G)\), then \(f_1 \in W_{t,x}^{3l_0,2l_0}(G)\). In this moment for the operator \(L_1\) and for the right hand \(f_1\) are fulfilled the conditions of the theorem. Hence
the problem (5)-(7) has a unique solution \( w \in W_{t,x}^{3(l_0+1),2(l_0+1)}(G) \) such that
\[
D_t^{i-1}w(0,x) = 0, \quad D_t^i w(0,x) = 0, \quad D_t^{i+1}w(T,x) = 0; \quad i = 1 + 3j, j = \frac{1}{l_0};
\]
almost everywhere in \( D \). Now \( D_t^3 u \equiv w \) almost everywhere in \( G \). Then
\[
w \in W_{t,x}^{3(l_0+1),2(l_0+1)}(G) \quad \text{from where} \quad u \in W_{t,x}^{0,2(l_0+2)}(G) \quad \text{and}
\]
\[
D_t^{i-1}u(0,x) = 0, \quad D_t^i u(0,x) = 0, \quad D_t^{i+1}u(T,x) = 0; \quad i = 1 + 3j, j = \frac{1}{l_0+1};
\]
almost everywhere in \( D \).

By integration by parts in the equality (4) from [3] we obtain
\[
\int_G \sum_{|\alpha|=1,|\beta|=2} a_{\alpha\beta}(x) D_x^\alpha u D_x^\beta \zeta dt dx = (f - [c(t,x) - \bar{C}] - k_3 D_t^3 u - k_4 D_t^4 u, \zeta) \forall \zeta \in \overline{C}_\infty(G).
\]
But \( C_0(G) \subset \overline{C}_\infty(G) \) if \( f \in W_{t,x}^{3(l_0+1),2(l_0+1)}(G) \) then \( f_2 \in W_{t,x}^{0,2l_0}(G) \).

Now from theorem 3 from [5] we conclude that \( u \in W_{t,x}^{0,2l_0}(G) \).

Finally the estimates from point 10.2 from [6] give that \( u \in W_{t,x}^{3(l_0+2),2(l_0+2)}(G) \).

The theorem is proved.

5. Classical solution

The considered domain fulfills a b-horn condition for \( b = (b_0, b_1, b_2, b_3, \ldots, b_n) \), \( b_i > 0, i = 0, \ldots, n; b_1 = b_2 = \ldots = b_n \);

For integer \( l \geq 1 \) and such that
\[
\frac{4}{3(l+1)} + \frac{1}{2} \left\{ \frac{1}{3(l+1)} + \frac{n}{2(l+1)} \right\} < 1
\]
from theorem 10.4 from [6] we obtain that the derivatives \( D_t^i u, i \leq 4 \) of the generalized solution \( u \) of the problem (1)-(3) are classical.

Also from the same theorem we obtain that for integer \( l \geq 1 \) and such that
\[
\frac{4}{2(l+1)} + \frac{1}{2} \left\{ \frac{1}{3(l+1)} + \frac{n}{2(l+1)} \right\} < 1
\]
is true then the derivatives \( D_t^i u, |\alpha| \leq 4 \) of the solution are classical. Now the last two inequalities reduce to \( l > \left[ \frac{13}{6} + \frac{n}{4} \right] - 1 \) and after integration by parts we conclude that the generalized solution of the problem (1)-(3) is a classical solution of this problem.
6. Example

Let \( n = 3, T = 1, A, C, R \) are positive constants and

\[ G = D \times (0,1), \Gamma = \partial D \times (0,1), \]
\[ D = \{(x_1, x_2, x_3) / x_1^2 + x_2^2 + x_3^2 < R^2\}, \]
\[ \Omega \in C^\infty[0,1], \Omega(0) = \Omega(1) = 0, f \in L^2(G). \]

Consider the problem

\[ \Omega(t)D_t^4 u + AD_t^3 u - D_x^4 u - D_x^2 u - D_x^3 u - Cu = f(t, x) \text{ in } G, \quad (14) \]
\[ D_x^2 u_{|\Gamma} = 0, \quad \alpha \leq 1 \quad (15) \]
\[ u(0, x) = 0, D_t u(0, x) = 0, D_t^2 u(1, x) = 0; \quad (16) \]

In this example

\[ k_4(t, x) \equiv \Omega(t), k_3(t, x) \equiv A, k_i(t, x) \equiv 0, \quad i = 1, 2, c(t, x) \equiv 0, \]
\[ a^{\alpha \beta}(x) \equiv 1, \alpha = \beta = (2, 0, 0), (0, 2, 0), (0, 0, 2) \text{ else } a^{\alpha \beta}(x) \equiv 0. \]

The equation (14) is a fourth order mixed type equation in \( G \cup \Gamma \) and parabolic on the bottoms of the cylinder. If \( f \in W^{3l,2l}(G) \) where \( l \geq 1 \) is a parameter and the constants \( A, C \) are sufficiently large and

\[ D_t^{i-1}f(0, x) = 0, D_t^i f(0, x) = 0, D_t^{i+1} f(T, x) = 0; \quad i = 1 + 3j, j = 0, l-1; \]

almost everywhere in \( D \), then the conditions of the above theorem are satisfied. Hence the problem (14)-(16) has a unique solution \( u \in W^{3l,2l}(G) \). If we take \( l=2 \) in the obtained condition, then the generalized solution of the problem (14)-(16) is a classical solution of this problem.

In the nonlocal case smooth solutions are considered also in [4]. For higher-order equations the present method of investigation is used in [1,2].

References


Department of Mathematics
Technical University, Plovdiv
25, Tsanko Dyustabanov Str.
4000 Plovdiv
BULGARIA

e-mail: g.p.paskalev@abv.bg
On isotropic compositions in pseudo–Weyl spaces

Ivan Badev

Abstract

Compositions in multi-dimensional spaces generalize lines of a net in two-dimensional spaces. Compositions are introduced by Norden, who defines Cartesian composition and adapted coordinate system to a composition. Yano studies the properties of compositions with structure affinor.

This work uses the structure affinor to study special isotropic composition in pseudo-Weyl space. In particular, invariant properties of Chebyshevian compositions are obtained with the methods of prolonged covariant differentiation and the coefficients of the derivative equation, introduced by Zlatanov. Also the pseudo-Riemannian space, which contains the special isotropic Cartesian composition, is found.

1. Introduction

Let \( W_n(g_{is}, P_k) \) be \( n \)-dimensional Weyl space with a fundamental tensor \( g_{is} \) and complementary covector \( P_k \). Denote with \( \nabla \) the covariant derivative and with \( \Gamma_{ki}^s \) the coefficient of connectedness. Then (1,p.137)

\[
\nabla_k g_{is} = 2P_k g_{is}, \quad \nabla_k g^{is} = -2P_k g^{is},
\] (1.1)

where \( g^{is} \) is the reciprocal to \( g_{is} \), \( (g^{ik} g_{is} = \delta^k_s) \) tensor.

Renormalize the principal tensor by the law \( \tilde{g}_{is} = \mu^2 g_{is} \), where \( \mu \) is a function of the point, to transform the additional covector as following (1,p.152)

\[
\tilde{P}_K = P_K + \partial_K \ln \mu.
\] (1.2)
The pseudo-quantity $A$, which transforms by $\tilde{A} = \sigma^r A \ (\sigma \text{ is the function of the point})$, is called satellite of $g_{is}$ with weight $\{r\}$. The prolonged covariant derivative $\overset{\circ}{\nabla}$ of satellite $A \ {r}$ of the principal tensor is [4]

$$\overset{\circ}{\nabla}_k = \nabla_k A - rP_k A. \quad (1.3)$$

From (1.1) and (1.2) it follows that

$$\overset{\circ}{\nabla}_k g^{is} = 0, \overset{\circ}{\nabla}_k g_{is} = 0. \quad (1.4)$$

Consider in $W_n$, the independent field of directions $v^i_\alpha (\alpha = 1, 2, \ldots, n)$. The renormalization $g_{is} v^i_\alpha v^s_\alpha = 1$ means that the vector $v^i_\alpha$ is satellite of $g_{is}$ with weight $\{-1\}$. The common covector $v^i_\alpha$ of $v^i_\alpha (v^i_\alpha v^s_\alpha = \delta^i_s)$ is a satellite with weight $\{-1\}$.

The derivative formulas are [4]:

$$\overset{\circ}{\nabla}_k g^{is} = 2P_k g^{is}, \overset{\circ}{\nabla}_k g_{is} = -2P_k g^{is}. \quad (1.5)$$

Any composition is completely defined by affinor $a^s_i$, which satisfies $a^k_s a^i_k = \delta^i_s$ and the condition for integrability [3]

$$a^\sigma_\beta \nabla_\alpha a^\delta_\sigma - a^\sigma_\alpha \nabla_\beta a^\delta_\sigma = 0.$$

The projecting affinors $a^s_i$ are given by

$$a^s_i = \frac{1}{2}(\delta^s_i + a^s_i), a^s_i = \frac{1}{2}(\delta^s_i - a^s_i). \quad (1.6)$$

These transform the vectors of their own position into themselves and the vectors of other positions into zero vectors.

Consider in $W_n$, composition $X_m \times X_{n-m}$ of the base manifolds $X_m$ and $X_{n-m}$. Through any point of $X_m \times X_{n-m} \in W_n$, two positions of the base manifolds $P(X_m)$ and $P(X_{n-m})$ pass through [3]. Any vector can by written as:

$$v^s = a^s_i v^i + a^s_i v^i = v^s + v^s, \quad \text{where} \quad v^s \in P(X_m), v^s \in P(X_{n-m}).$$

The pseudo-Weyl space $^1W_n(1_{g_{is}}, T_k)$, arising from elliptic space of Weyl $W_n(1_{g_{is}}, P_k)$, is with principal tensor [5]

$$1_{g_{is}} = v^i_1 v^s + \ldots + v^i_{n-1} v^s - v^i_1 v^s \quad (1.7)$$

and coefficient of connectedness $\Gamma^s_{ki}$. In this space there are fields of directions whose scalar squares are 1, -1, or 0. Let $\tilde{\nabla}_k$ be the symbol of prolonged covariant derivative generated by the connectedness $\Gamma^s_{ki}$. The derivative equations for $\tilde{\nabla}_k$ are:
On isotropic compositions in pseudo-Weyl spaces

\[ 1^\circ \nabla_k v^i = \frac{\beta}{\alpha} T^i_k, \quad 1^\circ \nabla_k v^i = -\frac{\alpha}{\beta} T^i_k. \]  

(1.8)

Also [5] demonstrates:

\[ T_{\alpha}^k + T_{\beta}^k = 0, (\alpha, \beta = 1, 2, \ldots, n - 1), \]

(1.9)

\[ T_{\alpha}^k - T_{\alpha}^n = 0, (\alpha = 1, 2, \ldots, n - 1), \]

(1.10)

(1.11)

Note that there is no summation along the bracketed indices.

Consider in \(1W_n(1g_{is}, T_k)\), the composition \(X_{n-1} \times X_1\) with affinor

\[ a^s_i = v_i n^s + v_i v^s + \frac{2}{2} v_i v^s + \frac{3}{3} v_i v^s + \ldots + \frac{n-1}{n-1} v^s, \]

(1.10)

and projective affinors

\[ a^s_i = \frac{1}{2} v_i n^s - v^s, \]

(1.11)

Then

\[ a^s_i v^s = v_i, a^s_i v^s = v_i, a^s_i v^s = v_i (\alpha = 2, 3, \ldots, n - 1); \]

(1.12)

\[ a^s_i (v^s + v^s) = v^s + v^s, a^s_i (v^s - v^s) = v^s - v^s. \]

(1.13)

The fields of directions \(v^s + v^s \in P(X_{n-1})\) and \(v^s - v^s \in P(X_1)\) are isotropic.

The composition \(X_{n-1} \times X_1 \in 1W_n\) is isotropic.

2. Invariant properties of isotropic compositions in pseudo-Weyl space.

The affinor \(a^s_i\) of \(X_{n-1} \times X_1 \in 1W_n\) and the projective affinors \(a^m_i\) allow to characterize the invariant properties of isotropic compositions. Composition \(X_m \times X_{n-m}\) is chebychevian \((Ch - Ch)\) if the position \(P(X_m)\) translates in parallel along \(P(X_{n-m})\) and the \(P(X_{n-m})\) translates in parallel along \(P(X_m)\) [2].
Theorem 1. Isotropic composition \( X_{n-1} \times X_1 \in W_n \) is \((Ch - Ch)\) iff the coefficients of \( T_k \) of the derivative equations satisfy:

\[
\frac{1}{n} T_k - \frac{n}{n} T_k \in P( X_{n-1} ), \frac{\alpha}{n} T_k - \frac{\alpha}{n} T_k \in P( X_{n-1} ), (\alpha = 2, 3, \ldots, n - 1). \tag{2.1}
\]

Proof. From [2], the composition \( X_{n-1} \times X_1 \in W_n \) is \((Ch - Ch)\) iff

\[
1\nabla_{(k} a_i^s \right) = 0.
\]

Since the affinor \( a_i^s \) has weight \( \{0\} \), this is equivalent to

\[
1\nabla_k a_i^s - 1\nabla_i a_k^s = 0. \tag{2.2}
\]

From here taking into account (1.10), (1.8) and (1.12) it follows that

\[
v^s_{1} \left[ \left( \frac{T_i - T_i}{2} \right) a^i_k - \left( \frac{T_k - T_k}{2} \right) \right] v_j + \left( \frac{T_i - T_i}{3} \right) a^i_k - \left( \frac{T_k - T_k}{3} \right) \right] v_j + \ldots +
\]

\[
\left( \frac{T_i - T_i}{n} \right) a^i_k - \left( \frac{T_k - T_k}{n} \right) \right] v_j + \ldots +
\]

Since the vectors \( \alpha \alpha \) and \( vi(\alpha = 1, 2, \ldots, n) \) are independent, from the last equality
On isotropic compositions in pseudo–Weyl spaces

\[
\left( \frac{\alpha}{T_i - T^i_k} \right) a^i_k - \left( \frac{\alpha}{T_k - T^k_i} \right) = 0,
\]
(2.3)

\[
\left( \frac{1}{T_i - T^i_k} \right) a^i_k - \left( \frac{1}{T_k - T^k_i} \right) = 0, (\alpha = 2, 3, \ldots, n - 1).
\]

Finally from (1.6) it follows that

\[
\frac{1}{T_k - T^k_i}, \frac{\alpha}{T_k - T^k_i} \in P( X_{n-1} ) \text{ for } (\alpha = 2, 3, \ldots, n - 1). \quad \Box
\]

The composition \( X_m \times X_{n-m} \) is \( X_{m - C \beta} \) if the position \( P( X_{n-m} ) \) translates in parallel along \( P( X_m ) \) [2].

**Theorem 2.** Isotropic composition \( X_{n-1} \times X_1 \in W_n \) is \( X_{m - C \beta} \)

iff for the coefficients \( T^k_k \) of the derivative equation satisfy:

\[
\frac{1}{T_k - T^k_i}, \frac{\alpha}{T_k - T^k_i} \in P( X_1 ), (\alpha = 2, 3, \ldots, n - 1).
\]
(2.4)

**Proof.** \( X_{n-1} \times X_1 \in W_n \) is \( X_{m - C \beta} \) iff [2]

\[
\frac{n}{m} a_{j^m}^{k^{m}} \frac{\alpha}{a_{i^m}^{\beta}} = 0.
\]
(2.5)

The projective affinor \( a_{i^m}^{\beta} \) has weight \( \{0\} \), so that (2.5) is equivalent to

\[
\frac{n}{m} a_{j^m}^{k^{m}} \frac{\alpha}{a_{i^m}^{\beta}} = 0.
\]
(2.6)

Now taking into account (1.10), (1.11) and (1.8) it follows that

\[
\frac{1}{T_j - T^j_j} + \frac{n}{m} a_{j^m}^{k^{m}} \left( \frac{n}{m} a_{i^m}^{\beta} \right) = 0,
\]
(2.7)

\[
\frac{\alpha}{T_j - T^j_j} + \frac{n}{m} a_{j^m}^{k^{m}} \left( \frac{\alpha}{a_{i^m}^{\beta}} \right) = 0, (\alpha = 2, 3, \ldots, n - 1).
\]

Finally (2.4) follows from (2.7) and (1.6). \( \Box \)

The composition \( X_m \times X_{n-m} \) is \( G - C \beta \), if \( P( X_m ) \) and \( P( X_{n-m} ) \) translate in parallel along \( P( X_m ) \). \( X_m \times X_{n-m} \) is \( G - C \beta \) iff [2]

\[
\frac{n}{m} a_{j^m}^{k^{m}} \frac{\alpha}{a_{i^m}^{\beta}} = 0.
\]
(2.8)
Theorem 3. Isotropic composition \( X_{n-1} \times X_1 \in W_n \) is \((G - Ch)\) iff:

\[
T_k - T_k, T_k - T_k, T_k - T_k \in P(\alpha), (\alpha = 2, 3, \ldots, n - 1).
\]  

(2.9)

**Proof.** Since \( a^n_{PB} \) has weight \( \{0\} \), (2.8) is equivalent to

\[
\alpha^o a^n = 0.
\]  

(2.10)

From (1.10), (1.11) and (1.8) equation (2.11) follows, which in turn implies (2.9).

\[
T_j - T_j + a^n_k \left( T_k - T_k \right) = 0,
\]  

(2.11)

Composition \( X_m \times X_{n-m} \) is \((X_m - C)\) if the position \( P(\alpha) \) translates in parallel along every line of \( X_m \). \( X_m \times X_{n-m} \) is \((X_m - C)\) iff [2]:

\[
a^n_m a^k_j \nabla a^n_i = 0.
\]  

(2.12)

Theorem 4. Isotropic composition \( X_{n-1} \times X_1 \in W_n \) is \((X_{n-1} - C)\) iff

\[
\alpha^o T_k - T_k = 0, (\alpha = 2, 3, \ldots, n - 1).
\]  

(2.13)

**Proof.** From (2.12), (1.10), (1.11) and (1.8) it follows that:

\[
T_k - T_k = 0,
\]  

(2.14)

from where (2.13) follows using (1.9).

3. Isotropic \( X_{n-1} \times X_1 \) composition in pseudo-Riemannian space.

Weyl space \( W_n( g_{is}, P_k) \), for which \( P_k = grad \) is Riemannian space [1,p.157].

The principal tensor \( g_{is} \) can be renormalized as to \( P_k = 0 \). The renormalized principal
tensor will be used for metric tensor in the Riemannian space. Consider the pseudo-riemannian space $1V_n(g_{is})$ with metric tensor (1.7) and derivative equations

$$1\nabla_k v^i_\alpha = T_k^i v^i_\alpha, \quad 1\nabla_k v^\alpha_i = -T_k^\alpha v^i_\beta.$$  

(3.1)

Affinor (1.10) defines isotropic composition $\mathbf{X}_{n-1} \times X_1 \in 1V_n$. From theorem 4 follows that the coefficients of (3.1) satisfy:

$$\alpha^\alpha T_k^n = \alpha^\alpha T_k^n = 0, (\alpha = 2, 3, \ldots, n - 1).$$  

(3.2)

Chose the net $(v, v, \ldots, v) \in 1V_n$ as a coordinate one. The metric tensor has the form:

$$g_{is} = f f \cos \omega, \quad i = 1, 2, n, \quad i = s$$


where $f = f(u, u, \ldots, u)$ and $\omega = \alpha(u, u, \ldots, u)$ is the angel between $v$ and $v$.

The coordinates of the vectors of the net and those of the metric tensor are:

$$v = (0, \ldots, 0, \frac{1}{f}, 0, \ldots, 0), \quad \alpha^\alpha v = (0, \ldots, 0, f, 0, \ldots, 0),$$

(3.3)

$$g_{ii} = f^2, \quad g_{ii} = (f^2)^{-1}, \quad g_{is} = 0, (i \neq s).$$

From the covariate derivative of $v$ and (3.1) it follows that

$$\beta^\beta v_i^i (\partial_k^i v^\alpha_i + \Gamma_k^i_{is} v^s_i) = \beta^\beta T_k^n.$$  

(3.4)

From (3.2), (3.3) and (3.4) for the parameters of the coordinate net it holds

$$\Gamma^{\alpha}_{kl} v^1_l = \Gamma^{\alpha}_{ki} v^n_r.$$  

(3.5)

The coefficients of connectedness can be written [1,p.133]

$$\Gamma^k_{ki} = \frac{1}{2} g^{kn} (\partial_i g_{sn} + \partial_s g_{in} - \partial_n g_{is}).$$  

(3.6)
From (3.5) and (3.6) the following obtains

\[ f(\hat{\partial}_i g_{k\alpha} - \hat{\partial}_\alpha g_{k1}) = f(\hat{\partial}_n g_{k\alpha} - \hat{\partial}_\alpha g_{kn}), (\alpha = 2, 3, \ldots, n - 1). \]

From here using (3.3) the functions \( f \) satisfy:

\[
\begin{align*}
\frac{\partial f}{\partial \alpha} \left( \frac{\partial f}{\partial n} \right)^2 &= 0, \\
\frac{\partial f}{\partial n} \left( \frac{\partial f}{\partial 1} \right)^2 &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial n} \left( \frac{\partial f}{\partial \alpha} \right)^2 &= \frac{\partial f}{\partial 1} \left( \frac{\partial f}{\partial n} \right)^2, (\alpha = 2, 3, \ldots, n - 1).
\end{align*}
\]

Thus

\[
\begin{align*}
f &= f(u, u), \\
f &= f(u, u) \\
f &= f(u, u, \ldots, u), (\alpha = 2, 3, \ldots, n - 1).
\end{align*}
\]

References

1. A. Norden. Affinely connected spaces, GRFML, Moscow, 1976 (in Russian)


Technical College J. Atanassov
Technical University
25 Tsanko Dyustabanov St.
4000 Plovdiv
BULGARIA
E-mail: ivanbadev@abv.bg
JOURNAL
OF THE TECHNICAL UNIVERSITY AT PLOVDIV
<FUNDAMENTAL SCIENCES AND APPLICATIONS>
ISSN 1310-8271

Editor-in-Chief
Peyo Stoilov

СПИСАНИЕ
НА ТЕХНИЧЕСКИЯ УНИВЕРСИТЕТ В ПЛОВДИВ
<ФУНДАМЕНТАЛНИ НАУКИ И ПРИЛОЖЕНИЯ>

ГЛАВЕН РЕДАКТОР
Пейо Стоилов

ИЗДАТЕЛСКО-ПОЛИГРАФИЧНО ЗВЕНО
Технически Университет - София, Филиал- Пловдив
4000 Пловдив, ул. Цанко Дюстабанов № 25