On some interpolation theorems for the multipliers of the Cauchy-Stiltjes type integrals

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Abstract

Let \( m_\alpha \) \((\alpha > 0)\) denote the set of all multipliers of the analytic functions in the unit disk \( D \), representable by a Cauchy-Stiltjes type integral \( \int_T \frac{1}{(1 - \zeta z)^\alpha} d\mu(\zeta) \).

Let the sequence \( a = \{a_k\}_{k \geq 1} \subset D \) and

\[ R_a f = \{f(a_k)\}_{k \geq 1}, \quad S_a f = \{f'(a_k)(1 - |a_k|^2)\}_{k \geq 1}. \]

In this paper we prove that if a sequence \( a = \{a_k\}_{k \geq 1} \subset D \) satisfies conditions of Newman-Carleson and Stoltz, then

\[ R_a f = b_v, \quad S_a f = l^1. \]

1 Introduction

Let \( D \) denote the unit disk in the complex plane and \( T \) - the unit circle. For \( 0 < p \leq \infty \) let \( H^p \) be the usual Hardy class [1].

Let \( M \) be the Banach space of all complex-valued Borel measures on \( T \) with the usual variation norm. For \( \alpha > 0 \), let \( F_\alpha \) denote the family of all functions \( g \) for which there exists \( \mu \in M \) such that

\[ g(z) = \int_T \frac{1}{(1 - \zeta z)^\alpha} d\mu(\zeta), \quad z \in D. \]
We note that $F_\alpha$ is a Banach space with the natural norm
\[ \|g\|_{F_\alpha} = \inf \{ \|\mu\| : \mu \in M \text{ such that } (1) \text{ holds} \} . \]

The family $F_\alpha$ was introduced in [2] and the following results were proved there:
If $f \in F_\alpha$, $g \in F_\beta$, then $f g \in F_{\alpha + \beta}$ ($\alpha > 0$, $\beta > 0$).

If $\alpha < \beta$, then $F_\alpha \subset F_\beta$.

Definition. Suppose that $f$ is holomorphic in $D$. Then $f$ is called a multiplier of $F_\alpha$ if $g \in F_\alpha \Rightarrow fg \in F_\alpha$.

In this paper some interpolation theorems for $m_1$ and $F_1$ obtained in [4] and [7] are generalized for $m_\alpha$ and $F_\alpha$.

2 Interpolation theorems

Definition. We say that a sequence $a = \{a_k\}_{k \geq 1} \subset D$ satisfies Newman-Carleson condition (condition $(N-C)$) if
\[ \delta(a) = \inf \left( \prod_{k \neq n} \left| \frac{a_k - a_n}{1 - \overline{a_k}a_n} \right| : n = 1, 2, \ldots \right) > 0. \]

Theorem 2. Let $\alpha > 0$ and the sequence $a = \{a_k\}_{k \geq 1}$ ($a_k \neq a_n$ if $k \neq n$ and $|a_k| \leq |a_{k+1}|$) satisfy conditions $(N-C)$ and $(S)$. Then
a) If $g \in F_\alpha$, then $(1 - a_k)^\alpha g(a_k) \in b_v$;

b) For each sequence $x = \{x_k\}_{k \geq 1} \in b_v$, there is a function $g$ in $F_\alpha$, such that $(1 - a_k)^\alpha g(a_k) = x_k$, $k = 1, 2, \ldots$.

Proof. a) Let $g \in F_\alpha$ and $f(z) = (1 - z)^\alpha g(z)$. If $n \geq [\alpha] + 1$ is a natural number, then
\[ \frac{f(z)}{(1 - z)^n} = \frac{f(z)}{(\zeta - z)^\alpha} \cdot \frac{1}{(\zeta - z)^{n-\alpha}} = g(z) \cdot \frac{1}{(\zeta - z)^{n-\alpha}} \in F_{\alpha + n - \alpha} = F_n \]
and from the proof of Theorem 1 follows
\[ \{f(a_k)\}_{k \geq 1} = \{(1 - a_k)^\alpha g(a_k)\}_{k \geq 1} \in b_v. \]
b) Let \( x = \{x_k\}_{k \geq 1} \in b_v \). From Theorem 1 it follows that there exists a function \( f \in m_\alpha \) such that \( f(a_k) = x_k, \ k = 1, 2, \ldots \). Then \( g(z) = \frac{f(z)}{(1-z)^\alpha} \in F_\alpha \) and \( (1-a_k)^\alpha g(a_k) = x_k, \ k = 1, 2, \ldots \).

The following theorem generalizes a result in [7].

**Theorem 3.** Let \( \alpha > 0 \) and the sequence \( a = \{a_k\}_{k \geq 1} \) satisfy conditions \( (N-C) \) and \( (S) \).

Then

a) If \( f \in m_\alpha \), then \( \{f'(a_k)(1-|a_k|^2)\}_{k \geq 1} \in l_1 \);

b) For each sequence \( x = \{x_k\}_{k \geq 1} \in l_1 \), there is a function \( f \) in \( m_\alpha \) such that \( f'(a_k)(1-|a_k|^2) = x_k, \ k = 1, 2, \ldots \).

**Proof.** a) Let \( f \in m_\alpha \) and \( n \geq \lceil \alpha \rceil + 1 \) is a natural number. Since \( f \in m_n \) then there exists a measure \( \mu \in M \) for which

**References**


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